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An Algebraic Derivation of the Eigenspaces Associated with an Ising-Like Spectrum of the Superintegrable Chiral Potts Model

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Abstract In terms of the $\mathfrak{sl}_2$ loop algebra and the algebraic Bethe-ansatz method, we derive the invariant subspace associated with a given Ising-like spectrum consisting of 2^r eigenvalues of the diagonal-to-diagonal transfer matrix of the superintegrable chiral Potts (SCP) model with arbitrary inhomogeneous parameters. We show that every regular Bethe eigenstate of the τ_2-model leads to an Ising-like spectrum and is an eigenvector of the SCP transfer matrix which is given by the product of two diagonal-to-diagonal transfer matrices with a constraint on the spectral parameters. We also show in a sector that the τ_2-model commutes with the $\mathfrak{sl}_2$ loop algebra, $L(\mathfrak{sl}_2)$, and every regular Bethe state of the τ_2-model is of highest weight. Thus, from physical assumptions such as the completeness of the Bethe ansatz, it follows in the sector that every regular Bethe state of the τ_2-model generates an $L(\mathfrak{sl}_2)$-degenerate eigenspace and it gives the invariant subspace, i.e. the direct sum of the eigenspaces associated with the Ising-like spectrum.

1 Introduction

The chiral Potts model [3, 5, 11, 31, 42], which is an N-state generalization of the two-dimensional Ising model, has been extensively studied from various points of view in recent years. The model is solvable in the sense that its Boltzmann weights satisfy the star-triangle relation to give a commutative family of transfer matrices [11]. In fact, the free energy, interfacial tension and order parameters of the model are exactly calculated in the thermodynamic limit [2, 8–10].

In the superintegrable case of the chiral Potts model, all the eigenvalues of the transfer matrix are grouped into sets of 2^r eigenvalues similar to those of free fermions. We

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call it the superintegrable chiral Potts (SCP) model and the set of eigenvalues an Ising-like spectrum [1, 2, 6–8, 41]. The Onsager algebra is powerful to derive the spectrum of the two-dimensional Ising model [18, 38, 39], in which a set of 2^r eigenvalues corresponds to a 2^r-dimensional irreducible representation of the algebra. The approach is extended to the $\mathbb{Z}_N$-symmetric quantum system corresponding to the SCP model [16, 18, 42]. However, in contrast to the Ising-case, the approach does not work enough to derive an exact form of the spectrum for $N \geqslant 3$. A derivation of the exact form is established by the approach [1, 6–8, 41] using functional relations among diagonal-to-diagonal transfer matrices of the SCP model [12]. There, the Ising-like spectrum is described by a polynomial, which we call the SCP polynomial. However, it is still nontrivial to define the SCP polynomial by an algebraic method.

In this paper, we present a method for constructing basis vectors of the direct sum of the eigenspaces associated with a given Ising-like spectrum of the transfer matrix of the SCP model in some sector. In short, we construct the invariant subspace of the Ising-like spectrum. First, by the algebraic Bethe-ansatz method, we show that every regular Bethe state of the τ_2-model is an eigenvector of the SCP transfer matrix. Here it is defined by the product of two diagonal-to-diagonal transfer matrices of the SCP model with a constraint on the spectral parameters. We shall define it in detail in Sect. 2.1. The τ_2-model is the integrable N-state spin chain corresponding to a nilpotent case of the cyclic L-operator [34, 35]; the transfer matrix constructed from the cyclic L-operators commutes with the transfer matrix of the chiral Potts model [13]. Secondly, we show in a sector that the τ_2-model has the symmetry of the $\mathfrak{sl}_2$ loop algebra, $L(\mathfrak{sl}_2)$, and also in the sector that every regular Bethe state of the τ_2-model is a highest weight vector of the symmetry. Thus, the degenerate eigenspaces are generated by regular Bethe eigenstates [25, 37] in the sector through the symmetry. Here we shall define regular Bethe states in Sect. 2.2. Thirdly, with some physical assumptions such as the completeness of the Bethe ansatz, we show that for the diagonal-to-diagonal transfer matrix of the SCP model the invariant subspace of the Ising-like spectrum associated with a regular Bethe state is given by the $L(\mathfrak{sl}_2)$-degenerate eigenspace of the τ_2-model generated by the same regular Bethe state.

We apply a generalization of the algebraic Bethe ansatz to the SCP transfer matrix with arbitrary inhomogeneous parameters, and do not use the functional relations among the transfer matrices [12]. The algebraic approach treats the SCP model and the τ_2-model in a unified way, which might be useful for calculating correlation functions for the model.

We reproduce the SCP polynomial as a kind of Drinfeld polynomial which characterizes the finite-dimensional highest weight representation of $L(\mathfrak{sl}_2)$ generated by the regular Bethe state. Here it is not necessarily irreducible [24]. For generic values of inhomogeneous parameters, however, the zeros of the polynomial should be distinct, and hence the highest weight representation should be irreducible. Thus, the SCP polynomial should be identified with the Drinfeld polynomial [15, 23, 24, 27].

The algebraic derivation of the invariant subspace associated with the Ising-like spectrum proves in the sector a previous conjecture that for the $\mathbb{Z}_N$-symmetric Hamiltonian the representation space of the Onsager algebra associated with an SCP polynomial should correspond to the $L(\mathfrak{sl}_2)$-degenerate eigenspace of the τ_2-model associated with the Drinfeld polynomial [37].

For the τ_2-model we shall show a Borel subalgebra symmetry of $L(\mathfrak{sl}_2)$ through a gauge transformation on the L-operators. In fact, it is known that every finite-dimensional irreducible representation of the Borel subalgebra is extended to that of the $\mathfrak{sl}_2$ loop algebra [14, 22]. We shall thus derive the $L(\mathfrak{sl}_2)$ symmetry of the τ_2-model in the general N-state case with inhomogeneous parameters in the paper. Previously, the symmetry has been shown for the odd N and homogeneous case [37]. The present result also proves the $L(\mathfrak{sl}_2)$ symmetry of the τ_2-model for the even N and homogeneous case. It thus proves the conjecture [4] that the proposed set of commuting operators forms the $L(\mathfrak{sl}_2)$ symmetry of the τ_2-model. The $L(\mathfrak{sl}_2)$ symmetry of the τ_2-model is closely related to that of the spin-1/2 XXZ spin chain at roots of unity [21, 24–26].

The article consists of the following: in Sect. 2, we introduce the SCP model and the τ_2-model. We review the algebraic Bethe-ansatz method for the τ_2-model [41] and the Yang-Baxter relation between the monodromy matrices of the two models [13]. In Sect. 3, generalizing the algebraic Bethe ansatz, we show that every regular Bethe eigenstate of the τ_2-model is an eigenvector of the SCP transfer matrix with a constraint on the spectral parameters. The expression of eigenvalues of the product of two diagonal-to-diagonal transfer matrices suggests the Ising-like spectrum to each of the two transfer matrices. In Sect. 4, we show in a sector the Borel subalgebra symmetry of the τ_2-model through a gauge transformation on the L-operators. It thus follows from [14, 22] that the τ_2-model has the $L(\mathfrak{sl}_2)$ symmetry. We also show in the sector that every regular Bethe state of the τ_2-model generates an irreducible highest weight representation of $L(\mathfrak{sl}_2)$, which gives the degenerate eigenspace associated with the regular Bethe state for the τ_2-model. We then formulate the conjecture that the diagonal-to-diagonal transfer matrix of the SCP model has the Ising-like spectrum in the $L(\mathfrak{sl}_2)$-degenerate eigenspace of the τ_2-model.

2 Models and Yang-Baxter Relations

2.1 The Chiral Potts Model and the Superintegrable Conditions

We briefly review the chiral Potts model [3, 5, 11] and its superintegrable point [1, 2, 6–8]. The model is defined on a two-dimensional square lattice with N-state local spins interacting along the edges. For two adjacent local spins σ_i and σ_j which take values in $\mathbb{Z}_N$, that is, $0, 1, \dots, N-1$, two edge-types of the Boltzmann weights $W_{pq}(\sigma_i - \sigma_j)$ and $\bar{W}_{pq}(\sigma_i - \sigma_j)$ are given as

$$W_{pq}(n) = W_{pq}(0)\prod_{j=1}^{n}\frac{\mu_p}{\mu_q}\frac{y_q - x_p\omega^j}{y_p - x_q\omega^j}, \qquad \bar{W}_{pq}(n) = \bar{W}_{pq}(0)\prod_{j=1}^{n}\mu_p\mu_q\frac{x_p\omega - x_q\omega^j}{y_q - y_p\omega^j},$$

where ω is an Nth root of unity. Here $p = (x_p, y_p, \mu_p)$ and $q = (x_q, y_q, \mu_q)$, which we call rapidities, are given on a Fermat curve defined by

$$kx_p^N = 1 - k'\mu_p^{-N}, \qquad ky_p^N = 1 - k'\mu_p^N, \quad k^2 + k'^2 = 1. \tag{2.1}$$

Note that both $W_{pq}(n)$ and $\bar{W}_{pq}(n)$ are functions of variable $n \in \mathbb{Z}_N$. The model is integrable in the sense that the Boltzmann weights satisfy the star-triangle relations [3, 11]. We also give the Fourier-transformed Boltzmann weight:

$$\widehat{W}_{pq}(n) = \sum_{m=0}^{N-1}\omega^{-nm}W_{pq}(m) = \widehat{W}_{pq}(0)\prod_{j=1}^{n}\frac{x_p\mu_p\omega - x_q\mu_q\omega^j}{y_q\mu_p - y_p\mu_q\omega^j}.$$

We introduce the S-operator [13, 17] to construct the monodromy matrix of the SCP model. Let Z and X be operators which have the action $Zv_\sigma = \omega^\sigma v_\sigma$ and $Xv_\sigma = v_{\sigma+1}$ for a standard basis $\{v_\sigma | \sigma \in \mathbb{Z}_N\}$ of the N-dimensional vector space $\mathbb{C}^N$. By using them and combining the Boltzmann weights, we define the S-operator $S(p,p';q,q')\in \mathrm{End}(\mathbb{C}^N\otimes\mathbb{C}^N)$ by

$$S(p,p';q,q')=\frac{1}{N^2}P_{\mathbb{C}^N\otimes\mathbb{C}^N}\sum_{\{n_i\}}w_{pp'qq'}(n_1,n_2,n_3,n_4)X^{n_1}Z^{n_2}X^{n_4}\otimes X^{-n_1}Z^{n_3}X^{-n_4},$$

$$w_{pp'qq'}(n_1,n_2,n_3,n_4)=\frac{\widehat{W}_{pq'}(n_1)}{W_{pq'}(0)}\frac{\bar{W}_{pq}(n_2)}{\bar{W}_{pq}(0)}\frac{\bar{W}_{p'q'}(n_3)}{\bar{W}_{p'q'}(0)}\frac{\widehat{W}_{p'q}(n_4)}{W_{p'q}(0)}, \tag{2.2}$$

where $P_{\mathbb{C}^N\otimes\mathbb{C}^N}$ is the standard permutation operator: $P_{\mathbb{C}^N\otimes\mathbb{C}^N}: v_\sigma\otimes v_\tau\mapsto v_\tau\otimes v_\sigma$. The action of the S-operator is extended to a tensor product $(\mathbb{C}^N)^{\otimes L}\otimes\mathbb{C}^N$, where the tensor product $(\mathbb{C}^N)^{\otimes L}$ is the quantum space describing an L-site spin chain and the last space $\mathbb{C}^N$ is an auxiliary space. We denote by $S_i(p,p';q,q')$ the S-operator acting on the ith component of $(\mathbb{C}^N)^{\otimes L}$ and auxiliary space $\mathbb{C}^N$ as the S-operator $S(p,p';q,q')$ and other components of $(\mathbb{C}^N)^{\otimes L}$ as the identity. Here we use the operators Z_i and X_i on $(\mathbb{C}^N)^{\otimes L}$ given by

$$Z_i=\mathrm{id}\otimes\cdots\otimes\overset{i}{\check{Z}}\otimes\cdots\otimes\mathrm{id},\qquad X_i=\mathrm{id}\otimes\cdots\otimes\overset{i}{\check{X}}\otimes\cdots\otimes\mathrm{id}.$$

We construct monodromy matrix $T(q_1,q_2;\{p,p'\})\in\mathrm{End}((\mathbb{C}^N)^{\otimes L}\otimes\mathbb{C}^N)$ and transfer matrix $t(q_1,q_2;\{p,p'\})\in\mathrm{End}((\mathbb{C}^N)^{\otimes L})$ as

$$T(q_1,q_2;\{p,p'\})=\prod_{i=1}^{L}S_i(p_i,p_i';q_1,q_2),\qquad t(q_1,q_2;\{p,p'\})=\mathrm{tr}_{\mathbb{C}^N}\big(T(q_1,q_2;\{p,p'\})\big), \tag{2.3}$$

where both p_i and p_i' are rapidities of the ith component of the quantum space $(\mathbb{C}^N)^{\otimes L}$ and q_1 and q_2 are those of the auxiliary space $\mathbb{C}^N$. The parameters q_1 and q_2 are called spectral parameters. Here the symbol $\{p,p'\}$ denotes the set of rapidities p_i and p_i' for $i=1,2,\dots,L$.

The transfer matrices satisfy the commutativity

$$t(q_1,q_2;\{p,p'\})t(r_1,r_2;\{p,p'\})=t(r_1,r_2;\{p,p'\})t(q_1,q_2;\{p,p'\}),$$

which is a result of the star-triangle relation [11]. Then the eigenvectors of the transfer matrix $t(q_1,q_2;\{p,p'\})$ are independent of the spectral parameters q_1 or q_2.

We call the transfer matrix $t(q_1,q_2;\{p,p'\})$ the row-to-row transfer matrix of the chiral Potts model since the Boltzmann weight $w_{pp'qq'}(n_1,n_2,n_3,n_4)$ is considered as that of a vertex model. The row-to-row transfer matrix $t(q_1,q_2;\{p,p'\})$ is given by the product of two types of diagonal-to-diagonal transfer matrices $T_{\mathrm{D}}(x_{q_1},y_{q_1})$ and $\hat{T}_{\mathrm{D}}(x_{q_2},y_{q_2})$ which are defined by

$$T_{\mathrm{D}}(x_q,y_q)_\sigma^{\sigma'}=\prod_{i=1}^{L}\frac{W_{p_i'q}(\sigma_i-\sigma_i')}{W_{p_i'q}(0)}\frac{\bar{W}_{p_{i+1}q}(\sigma_{i+1}-\sigma_i')}{\bar{W}_{p_{i+1}q}(0)},$$

$$\hat{T}_{\mathrm{D}}(x_q,y_q)_\sigma^{\sigma'}=\prod_{i=1}^{L}\frac{\bar{W}_{p_i'q}(\sigma_i-\sigma_i')}{\bar{W}_{p_i'q}(0)}\frac{W_{p_{i+1}q}(\sigma_i-\sigma_{i+1}')}{W_{p_{i+1}q}(0)},$$

where the periodic boundary conditions $\sigma_{L+1}=\sigma_1$ and $p_{L+1}=p_1$ are imposed. The diagonal-to-diagonal transfer matrices are diagonalized by a pair of invertible matrices U

and V, which are independent of the parameter q, as $U^{-1}T_{\mathrm{D}}(x_q, y_q)V = \Lambda(x_q, y_q)$ and $V^{-1}\hat{T}_{\mathrm{D}}(x_q, y_q)U = \hat{\Lambda}(x_q, y_q)$.

Let us now discuss the superintegrable case. When rapidities p and p' satisfy the conditions $x_p = y_{p'}$, $y_p = x_{p'}$, $\mu_p = \mu_{p'}^{-1}$, we denote the rapidity p' by $\bar{p}$.

Definition 2.1 We call the chiral Potts model superintegrable, if rapidities $\{p, p'\}$ satisfy the conditions $p'_i = \bar{p}_i$ for all i, that is, $x_{p_i} = y_{p'_i}$, $y_{p_i} = x_{p'_i}$, $\mu_{p_i} = \mu_{p'_i}^{-1}$ $(1 \leqslant i \leqslant L)$ [6–8].

In the superintegrable case, we denote by $T(q_1, q_2; \{p\})$ and $t(q_1, q_2; \{p\})$ the monodromy matrix and the row-to-row transfer matrix of the chiral Potts model, respectively. That is, we express $T(q_1, q_2; \{p\}) = T(q_1, q_2; \{p, \bar{p}\})$ and $t(q_1, q_2; \{p\}) = t(q_1, q_2; \{p, \bar{p}\})$. Hereafter we call the row-to-row transfer matrix $t(q_1, q_2; \{p\})$ the SCP transfer matrix, briefly. We also call $T(q_1, q_2; \{p\})$ the SCP monodromy matrix.

2.2 The τ_2-Model and the Algebraic Bethe Ansatz

Let us now introduce an integrable N-state spin chain whose transfer matrix commutes with the SCP transfer matrix. We introduce the cyclic L-operator $\mathcal{L}(z; p, p') \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^N)$ [34, 35] by

$$\mathcal{L}(z; p, p') = \begin{pmatrix} -y_p y_{p'} z + \mu_p \mu_{p'} Z & -z(y_p - x_{p'}\mu_p\mu_{p'}Z)X \\ X^{-1}(y_{p'} - x_p\mu_p\mu_{p'}Z) & 1 - x_p x_{p'}\mu_p\mu_{p'} z\omega Z \end{pmatrix}. \tag{2.4}$$

In the same way as the S-operator, the action of the L-operator $\mathcal{L}(z; p, p')$ is extended to the tensor product $\mathbb{C}^2 \otimes (\mathbb{C}^N)^{\otimes L}$ where the space $\mathbb{C}^2$ is another auxiliary space; we denote by $\mathcal{L}_i(z; p, p')$ the L-operator acting on the auxiliary space and ith component of the quantum space $(\mathbb{C}^N)^{\otimes L}$ as the L-operator $\mathcal{L}(z; p, p')$ and other components of $(\mathbb{C}^N)^{\otimes L}$ as the identity. The following properties give the reason why the chiral Potts model is considered as a descendant of the six-vertex model [13]:

Proposition 2.2 *The L-operators $\mathcal{L}_i(z) = \mathcal{L}_i(z; p, p')$ satisfy a Yang-Baxter relation*

$$R(z/w)\big(\mathcal{L}_i(z) \otimes \mathrm{id}_{\mathbb{C}^2}\big)\big(\mathrm{id}_{\mathbb{C}^2} \otimes \mathcal{L}_i(w)\big) = \big(\mathrm{id}_{\mathbb{C}^2} \otimes \mathcal{L}_i(w)\big)\big(\mathcal{L}_i(z) \otimes \mathrm{id}_{\mathbb{C}^2}\big)R(z/w) \tag{2.5}$$

with the R-matrix defined by

$$R(z) = \begin{pmatrix} 1 - z\omega & 0 & 0 & 0 \\ 0 & \omega(1-z) & (1-\omega)z & 0 \\ 0 & 1-\omega & 1-z & 0 \\ 0 & 0 & 0 & 1 - z\omega \end{pmatrix}, \tag{2.6}$$

and another Yang-Baxter relation

$$S_{ij}(p, p'; q, q')\mathcal{L}_i(z; p, p')\mathcal{L}_j(z; q, q') = \mathcal{L}_j(z; q, q')\mathcal{L}_i(z; p, p')S_{ij}(p, p'; q, q'), \tag{2.7}$$

where $S_{ij}(p, p'; q, q')$ is the S-operator acting on the ith and jth components of the quantum space $(\mathbb{C}^N)^{\otimes L}$ as the S-operator $S(p, p'; q, q')$ and other components of $(\mathbb{C}^N)^{\otimes L}$ as the identity.

At the superintegrable point, the cyclic L-operator $\mathcal{L}_i(z; p_i, p_i')$ is reduced to

$$\mathcal{L}_i(z; p_i, \bar{p}_i) = \begin{pmatrix} -t_{p_i}z + Z_i & -y_{p_i}z(1-Z_i)X_i \\ x_{p_i}X_i^{-1}(1-Z_i) & 1-t_{p_i}z\omega Z_i \end{pmatrix}, \tag{2.8}$$

where we have defined $t_p = x_p y_p$. We introduce the monodromy matrix $\mathcal{T}(z; \{p\}) \in \mathrm{End}(\mathbb{C}^2 \otimes (\mathbb{C}^N)^{\otimes L})$ and the transfer matrix $\tau(z; \{p\}) \in \mathrm{End}((\mathbb{C}^N)^{\otimes L})$ by

$$\mathcal{T}(z; \{p\}) = \prod_{i=1}^{L} \mathcal{L}_i(z; p_i, \bar{p}_i) =: \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}, \qquad \tau(z; \{p\}) = \mathrm{tr}_{\mathbb{C}^2}\big(\mathcal{T}(z; \{p\})\big). \tag{2.9}$$

Here we have also defined operators $A(z), B(z), C(z), D(z) \in \mathrm{End}((\mathbb{C}^N)^{\otimes L})$. The spin chain described by the transfer matrix $\tau(z; \{p\})$ is called the τ_2-model [4, 10, 40]. We remark that the original τ_2-model is defined in terms of the cyclic L-operator $\mathcal{L}(z; p, p')$ (2.4).

Proposition 2.3 *The monodromy matrix $\mathcal{T}(z; \{p\})$ satisfies a Yang-Baxter relation*

$$\begin{aligned} &R(z/w)\big(\mathcal{T}(z; \{p\}) \otimes \mathrm{id}_{\mathbb{C}^2}\big)\big(\mathrm{id}_{\mathbb{C}^2} \otimes \mathcal{T}(w; \{p\})\big) \\ &\quad = \big(\mathrm{id}_{\mathbb{C}^2} \otimes \mathcal{T}(w; \{p\})\big)\big(\mathcal{T}(z; \{p\}) \otimes \mathrm{id}_{\mathbb{C}^2}\big)R(z/w), \end{aligned} \tag{2.10}$$

where $R(z)$ is the R-matrix defined in (2.6).

The Yang-Baxter relation (2.10) gives the commutativity $\tau(z; \{p\})\tau(w; \{p\}) = \tau(w; \{p\})\tau(z; \{p\})$. Hence the eigenvectors of the transfer matrix $\tau(z; \{p\})$ are independent of the parameter z. The relation also produces relations among operators $A(z), B(z), C(z)$ and $D(z)$ [30]. In the next section, we need more general relations, which are collected in Lemma A.1. By using the relations, the algebraic Bethe-ansatz method is readily applicable to the transfer matrix $\tau(z; \{p\})$ [41].

Let $|0\rangle$ be the reference state $v_0 \otimes v_0 \otimes \cdots \otimes v_0$. It has the following properties:

$$\begin{aligned} A(z)|0\rangle &= a(z)|0\rangle = \prod_{n=1}^{L}(1 - t_{p_n}z)|0\rangle, \\ D(z)|0\rangle &= d(z)|0\rangle = \prod_{n=1}^{L}(1 - t_{p_n}z\omega)|0\rangle, \\ C(z)|0\rangle &= 0, \end{aligned}$$

for arbitrary z.

Proposition 2.4 *Let $\{z_i | i = 1, 2, \ldots, M\}$ be a solution of the Bethe equations*:

$$a(z_i) \prod_{\substack{j=1 \\ j(\neq i)}}^{M} f(z_i/z_j) = d(z_i) \prod_{\substack{j=1 \\ j(\neq i)}}^{M} f(z_j/z_i), \tag{2.11}$$

where $f(z) = (z-\omega)/(z-1)\omega$. Then, vector $|M\rangle = B(z_1)B(z_2)\cdots B(z_M)|0\rangle$ gives an eigenvector of the transfer matrix $\tau(z; \{p\})$:

$$\tau(z; \{p\})|M\rangle = \left(a(z)\prod_{i=1}^{M}\omega f(z/z_i) + d(z)\prod_{i=1}^{M}\omega f(z_i/z)\right)|M\rangle. \tag{2.12}$$

The vector $|M\rangle$ is referred to as a Bethe state.

If solutions of the Bethe equations (2.11) are non-zero, finite and distinct, we call them regular [25]. If $\{z_i | i = 1, 2, \ldots, R\}$ is a regular solution of the Bethe equations, we call the Bethe state $B(z_1)B(z_2)\cdots B(z_R)|0\rangle$ regular, and denote it by $|R\rangle$.

2.3 Commutativity of Transfer Matrices

As a consequence of the relation (2.7), we obtain the Yang-Baxter relation between the monodromy matrices $\mathcal{T}(z; \{p\})$ and $T(q_1, q_2; \{p\})$, by which we shall generalize the algebraic Bethe-ansatz method.

Proposition 2.5 *The monodromy matrices $\mathcal{T}(z; \{p\})$ and $T(q_1, q_2; \{p\})$ satisfy*

$$\begin{aligned}&\mathcal{L}(z; q_1, q_2)\big(\mathcal{T}(z; \{p\}) \otimes \mathrm{id}_{\mathbb{C}^N}\big)\big(\mathrm{id}_{\mathbb{C}^2} \otimes T(q_1, q_2; \{p\})\big)\\ &\quad = \big(\mathrm{id}_{\mathbb{C}^2} \otimes T(q_1, q_2; \{p\})\big)\big(\mathcal{T}(z; \{p\}) \otimes \mathrm{id}_{\mathbb{C}^N}\big)\mathcal{L}(z; q_1, q_2).\end{aligned} \tag{2.13}$$

Here the cyclic L-operator $\mathcal{L}(z; q_1, q_2)$ defined in (2.4) is considered as a $2N \times 2N$ matrix acting on the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^N$ of the auxiliary spaces. As a corollary, the transfer matrix $\tau(z; \{p\})$ commutes with the transfer matrix $t(q_1, q_2; \{p\})$.

Thanks to the commutativity of the two transfer matrices $\tau(z; \{p\})$ and $t(q_1, q_2; \{p\})$, they may have a set of common eigenvectors. For the τ_2-model, we have obtained the eigenstates through the algebraic Bethe-ansatz method. If a given Bethe eigenvector of the τ_2-model has a non-degenerate eigenvalue of $\tau(z; \{p\})$, then it also becomes an eigenvector of the SCP transfer matrix $t(q_1, q_2; \{p\})$. However, in Sect. 4, we shall show in a sector that the transfer matrix $\tau(z; \{p\})$ of the τ_2-model has degenerate eigenvectors with respect to the $\mathfrak{sl}_2$ loop algebra and hence not all the Bethe states of the τ_2-model are necessarily eigenvectors of $t(q_1, q_2; \{p\})$.

3 Spectrum of the Superintegrable Chiral Potts Model

We shall show in this section that, if the spectral parameters q_1 and q_2 satisfy the condition $q_2 = \bar{q}_1(s) = (y_{q_1}, x_{q_1}\omega^s, \mu_{q_1}^{-1})$, every regular Bethe eigenstate of $\tau(z; \{p\})$ is an eigenstate of the SCP transfer matrix $t(q_1, q_2; \{p\})$.

3.1 Algebraic Bethe-Ansatz Method for the SCP Transfer Matrix

First, we give a fundamental relation, generalizing the standard algebraic Bethe-ansatz method.

Proposition 3.1 *Let B_i, A_i and D_i denote $B(z_i)$, $A(z_i)$ and $D(z_i)$, respectively. Let $T_\tau^{\tau'}$, $(\tau, \tau' \in \mathbb{Z}_N)$ denote the operator-valued entries of the SCP monodromy matrix $T(q_1, q_2; \{p\})$. By setting $q_1 = q = (x_q, y_q, \mu_q)$ and $q_2 = \bar{q}(s) = (y_q, x_q\omega^s, \mu_q^{-1})$ $(s = 0, 1, \ldots, N-1)$ we have*

$$\begin{aligned}&B_1 \cdots B_n T_\tau^{\tau'}|0\rangle\\ &\quad = \sum_{\substack{\{i_\ell\},\{j_\ell\},\{k_\ell\}\\ n_B+n_A+n_D=n}} c_n^{\tau'\tau}(\{i_\ell\}; \{j_\ell\}; \{k_\ell\}) a(z_{j_1})\cdots a(z_{j_{n_A}}) d(z_{k_1})\cdots d(z_{k_{n_D}}) T_{\tau+n_A}^{\tau'-n_D} B_{i_1}\cdots B_{i_{n_B}}|0\rangle.\end{aligned} \tag{3.1}$$

Here $\{i_\ell\}, \{j_\ell\}$ *and* $\{k_\ell\}$ *are such disjoint subsets of the index set* $\Sigma_n = \{1, 2, \ldots, n\}$ *that the numbers of elements of the subsets denoted by* $\sharp\{i_\ell\} = n_B$, $\sharp\{j_\ell\} = n_A$ *and* $\sharp\{k_\ell\} = n_D$, *respectively, satisfy the condition* $n_B + n_A + n_D = n$, *and the coefficients* $c_n^{\tau'\tau}(\{i_\ell\}; \{j_\ell\}; \{k_\ell\})$ *are given by*

$$\begin{aligned} & c_n^{\tau'\tau}(\{i_\ell\}; \{j_\ell\}; \{k_\ell\}) \\ & \quad = \prod_{\ell=1}^{n_B} \frac{1}{\mu_{\tau'\tau}(z_{i_\ell})} \prod_{\ell=1}^{n_A} \frac{\nu_{\tau+\ell}(z_{j_\ell})}{\mu_{\tau',\tau+\ell-1}(z_{j_\ell})} \prod_{\ell=1}^{n_D} \frac{-\nu_{\tau'-\ell+1}(z_{k_\ell})}{\mu_{\tau'-\ell+1,\tau}(z_{k_\ell})} \prod_{\substack{i\in\{i_\ell\} \\ j\in\{j_\ell\}}} \omega f_{ji} \prod_{\substack{i\in\{i_\ell\} \\ k\in\{k_\ell\}}} \omega f_{ik} \prod_{\substack{j\in\{j_\ell\} \\ k\in\{k_\ell\}}} \omega f_{jk}, \end{aligned}$$

with $\mu_{\tau'\tau}(z)$, $\nu_\tau(z)$ *and* f_{ij} *defined by*

$$\mu_{\tau'\tau}(z) = \frac{(t_q z - 1)(t_q z \omega^s - 1)\omega^{\tau'}}{(t_q z \omega^{\tau'} - 1)(t_q z \omega^{\tau+1} - 1)}, \qquad \nu_\tau(z) = \frac{y_q z(1 - \omega^\tau)}{t_q z \omega^\tau - 1}, \qquad f_{ij} = \frac{z_i - z_j \omega}{(z_i - z_j)\omega}. \tag{3.2}$$

A proof of the relation is presented in the next subsection. The point of the proof is to arrange the product $B_1 \cdots B_n T_\tau^{\tau'}$ into the order $TBADC$, which is possible by the help of the relations (A.1) and (3.10). On the reference state $|0\rangle$, the terms with the operator $C(z_i)$ vanish and the operators $A(z_i)$ and $D(z_i)$ are replaced by the factors $a(z_i)$ and $d(z_i)$, respectively.

Next, we apply the transfer matrix $t(q, \bar{q}(s); \{p\})$ to the reference state $|0\rangle$. It is directly shown that the reference state $|0\rangle$ is an eigenvector of the transfer matrix $t(q, \bar{q}(s); \{p\})$ [6],

$$\begin{aligned} t(q, \bar{q}(s); \{p\})|0\rangle &= \sum_{\tau=0}^{N-1} \lambda_\tau |0\rangle \\ &= N^L \left(\prod_{n=1}^{L} \frac{x_{p_n} - x_q}{x_{p_n}^N - x_q^N} \frac{y_{p_n} - y_q}{y_{p_n}^N - y_q^N} \right) \sum_{\tau=0}^{N-1} \left(\prod_{n=1}^{L} \frac{t_{p_n}^N - t_q^N}{t_{p_n} - t_q \omega^\tau} \right) \omega^{\tau L} |0\rangle, \end{aligned} \tag{3.3}$$

where $t_{p_n} = x_{p_n} y_{p_n}$ and $t_q = x_q y_q$. Here we define λ_τ by relation (3.3).

Let $\{z_i | i = 1, 2, \ldots, R\}$ be a regular solution of the Bethe equations (2.11) and extract the term $T_\tau^{\tau'} B_1 \cdots B_R |0\rangle$ from the right-hand side of the relation (3.1). Then we see how the operator $T_\tau^{\tau'}$ acts on the regular Bethe state $|R\rangle$. By setting $\tau = \tau'$ and taking the sum on τ, it follows that the Bethe state $|R\rangle$ is an eigenvector of the SCP transfer matrix $t(q, \bar{q}(s); \{p\})$.

Theorem 3.2 *Every regular Bethe state* $|R\rangle$ *is an eigenvector of the transfer matrix* $t(q, \bar{q}(s); \{p\})$ *with* $q = (x_q, y_q, \mu_q)$ *and* $\bar{q}(s) = (y_q, x_q \omega^s, \mu_q^{-1})$,

$$\begin{aligned} & t(q, \bar{q}(s); \{p\})|R\rangle \\ & \quad = \sum_{\tau=0}^{N-1} \lambda_\tau \left(\prod_{i=1}^{R} \mu_{\tau\tau}(z_i) \right) |R\rangle \\ & \quad = N^L \left(\prod_{n=1}^{L} \frac{x_{p_n} - x_q}{x_{p_n}^N - x_q^N} \frac{y_{p_n} - y_q}{y_{p_n}^N - y_q^N} \right) \sum_{\tau=0}^{N-1} \left(\prod_{n=1}^{L} \frac{t_{p_n}^N - t_q^N}{t_{p_n} - t_q \omega^\tau} \right) \frac{\omega^{\tau(L+R)} F(t_q) F(t_q \omega^s)}{F(t_q \omega^\tau) F(t_q \omega^{\tau+1})} |R\rangle, \end{aligned} \tag{3.4}$$

where $F(t)$ is a polynomial in t defined by $F(t)=\prod_{i=1}^{R}(1-tz_i)$ and $\{z_i|i=1,\dots,R\}$ is a regular solution of the Bethe equations (2.11).

Proof By taking the sum of the left-hand side of the relation (3.1) with $\tau=\tau'$ over $\tau=0,1,\dots,N-1$ and using the result (3.3), we obtain

$$\sum_{\tau=0}^{N-1}\left(\prod_{i=1}^{R}\mu_{\tau\tau}(z_i)\right)B_1\cdots B_R T_\tau^\tau|0\rangle=\sum_{\tau=0}^{N-1}\lambda_\tau\left(\prod_{i=1}^{R}\mu_{\tau\tau}(z_i)\right)B_1\cdots B_R|0\rangle.$$

On the other hand, from the right-hand side of the relation (3.1), we have

$$\begin{aligned}
&\sum_{\tau=0}^{N-1}\left(\prod_{i=1}^{R}\mu_{\tau\tau}(z_i)\right)B_1\cdots B_R T_\tau^\tau|0\rangle\\
&=\sum_{\tau=0}^{N-1}\sum_{\substack{\{i_\ell\},\{j_\ell\},\{k_\ell\}\\ n_B+n_A+n_D=R}}\prod_{p=1}^{n_A}\frac{\mu_{\tau\tau}(z_{j_p})\nu_{\tau+p}(z_{j_p})}{\mu_{\tau,\tau+p-1}(z_{j_p})}\prod_{q=1}^{n_D}\frac{-\mu_{\tau\tau}(z_{k_q})\nu_{\tau-q+1}(z_{k_q})}{\mu_{\tau-q+1,\tau}(z_{k_q})}\\
&\quad\times\left(\prod_{\substack{i\in\{i_\ell\}\\ j\in\{j_\ell\}}}\omega f_{ji}\prod_{\substack{i\in\{i_\ell\}\\ k\in\{k_\ell\}}}\omega f_{ik}\prod_{\substack{j\in\{j_\ell\}\\ k\in\{k_\ell\}}}\omega f_{jk}\right)a_{j_1}\cdots a_{j_{n_A}}d_{k_1}\cdots d_{k_{n_D}}T_{\tau+n_A}^{\tau-n_D}B_{i_1}\cdots B_{i_{n_B}}|0\rangle\\
&=\sum_{m=0}^{R}\sum_{\substack{\{i_\ell\}\\ n_B=m}}\left(\prod_{i\in\Sigma_R\backslash\{i_\ell\}}a_i\right)\sum_{\tau=0}^{N-1}\sum_{\substack{\{j_\ell\},\{k_\ell\}\\ n_A+n_D=R-m}}\prod_{p=1}^{n_A}\frac{\mu_{\tau\tau}(z_{j_p})\nu_{\tau+p}(z_{j_p})}{\mu_{\tau,\tau+p-1}(z_{j_p})}\prod_{q=1}^{n_D}\frac{-\mu_{\tau\tau}(z_{k_q})\nu_{\tau-q+1}(z_{k_q})}{\mu_{\tau-q+1,\tau}(z_{k_q})}\\
&\quad\times\omega^{m(R-m)+n_An_D}\prod_{r=1}^{n_B}\prod_{p=1}^{n_A}\prod_{q=1}^{n_D}f_{j_pi_r}f_{k_qi_r}f_{k_qj_p}T_{\tau+n_A}^{\tau-n_D}B_{i_1}\cdots B_{i_m}|0\rangle\\
&=\sum_{m=0}^{R}\sum_{\substack{\{i_\ell\}\\ n_B=m}}\left(\prod_{i\in\Sigma_R\backslash\{i_\ell\}}a_i\right)\sum_{\tau=0}^{N-1}\sum_{\substack{\{j_\ell\},\{k_\ell\}\\ n_A+n_D=R-m}}\omega^{m(R-m)+n_An_D}\\
&\quad\times\prod_{p=1}^{n_A}\frac{\mu_{\tau+n_D,\tau+n_D}(z_{j_p})}{\mu_{\tau+n_D,\tau+n_D+p-1}(z_{j_p})}\nu_{\tau+n_D+p}(z_{j_p})\prod_{q=1}^{n_D}\frac{-\mu_{\tau+n_D,\tau+n_D}(z_{k_q})}{\mu_{\tau+n_D-q+1,\tau+n_D}(z_{k_q})}\nu_{\tau+n_D-q+1}(z_{k_q})\\
&\quad\times\prod_{r=1}^{n_B}\prod_{p=1}^{n_A}\prod_{q=1}^{n_D}f_{j_pi_r}f_{k_qi_r}f_{k_qj_p}T_{\tau+R-m}^{\tau}B_{i_1}\cdots B_{i_m}|0\rangle\\
&=\sum_{\tau=0}^{N-1}T_\tau^\tau B_1\cdots B_R|0\rangle=t(q,\bar q(s);\{p\})|R\rangle.
\end{aligned}\tag{3.5}$$

In the second equality, we have calculated as

$$\left(\prod_{i\in\{i_\ell\}}\prod_{j\in\{j_\ell\}}\prod_{k\in\{k_\ell\}} f_{ik}f_{jk}\right) d_{k_1}\cdots d_{k_{n_D}}$$
$$=\prod_{k\in\{k_\ell\}}\left(d_k\prod_{i\in\{i_\ell\}\cup\{j_\ell\}} f_{ik}\right)=\prod_{k\in\{k_\ell\}}\left(a_k\prod_{i\in\{i_\ell\}\cup\{j_\ell\}} f_{ki}\right)=\left(\prod_{i_\ell\in\{i_\ell\}}\prod_{j_\ell\in\{j_\ell\}}\prod_{k_\ell\in\{k_\ell\}} f_{ki}f_{kj}\right) a_{k_1}\cdots a_{k_{n_D}}$$

where we have employed the identity in Lemma B.3 which is derived from the Bethe equations (2.11). The unwanted terms $m\neq R$ in (3.5) have been canceled out because of the identity in Lemma B.4. □

3.2 The Ising-Like Spectrum Consisting of 2^r Eigenvalues

From the expression of eigenvalues of the SCP transfer matrix $t(q,\bar{q}(s);\{p\})$ we can derive eigenvalues of the diagonal-to-diagonal transfer matrices $T_{\rm D}(x_q,y_q)$ and $\hat{T}_{\rm D}(x_q,y_q)$. From the discussion similar to [8], the set of eigenvalues in the invariant subspace containing a given regular Bethe state $|R\rangle$ are given in the following forms:

$$\begin{aligned}
\Lambda(x_q,y_q)&=N^{\frac{L}{2}}\left(\prod_{n=1}^{L}\frac{x_{p_n}-x_q}{x_{p_n}^N-x_q^N}\right)x_q^{P_a}y_q^{P_b}\mu_q^{-NP_c}F(t_q)G(\mu_q^{-N}),\\
\hat{\Lambda}(x_q,y_q)&=N^{\frac{L}{2}}\left(\prod_{n=1}^{L}\frac{y_{p_n}-x_q}{y_{p_n}^N-x_q^N}\right)x_q^{P_a}y_q^{P_b}\mu_q^{-NP_c}F(t_q)\hat{G}(\mu_q^{-N}).
\end{aligned}\tag{3.6}$$

Here P_a and P_b are integers satisfying $P_a+P_b\equiv -L-R \mod N$, and we recall $F(t_q)=\prod_{i=1}^{R}(1-t_qz_i)$. $G(\mu_q^N)$ and $\hat{G}(\mu_q^N)$ are polynomials in μ_q^N satisfying $G(\mu_q^N)=\text{const.}\hat{G}(\mu_q^N)$.

From the relation $t(q,\bar{q}(s);\{p\})=T_{\rm D}(x_q,y_q)\hat{T}_{\rm D}(y_q,x_q\omega^s)$, the product $G(\mu_q^{-N})\hat{G}(\mu_q^N)$ is given by

$$G(\mu_q^{-N})\hat{G}(\mu_q^N)=P_{\rm SCP}(t_q^N):=\sum_{\tau=0}^{N-1}\left(\prod_{n=1}^{L}\frac{t_{p_n}^N-t_q^N}{t_{p_n}-t_q\omega^\tau}\right)\frac{\omega^{\tau(L+R)}}{F(t_q\omega^\tau)F(t_q\omega^{\tau+1})}.\tag{3.7}$$

Here $P_{\rm SCP}(t_q^N)$ is a polynomial in t_q^N of degree at most $\lfloor\frac{L(N-1)-2R}{N}\rfloor$; the Bethe equations (2.11) correspond to the pole-free condition. We call the polynomial $P_{\rm SCP}(\zeta)$ the SCP polynomial. We remark that, in our result, only the case $P_b=0$ appears. The relation $k^2t_q^N=1-k'(\mu_q^N+\mu_q^{-N})+k'^2$ tells us that the polynomial $P_{\rm SCP}(t_q^N)$ is regarded as a Laurent polynomial in μ_q^N of degree $r=\deg P_{\rm SCP}(\zeta)$ whose zeros occur in reciprocal pairs. Then, by denoting the $2r$ zeros by $\{w_i^{\pm1}\}$, we have 2^r solutions for $G(\mu_q^N)$ and $\hat{G}(\mu_q^N)$ in the forms

$$G(\mu_q^N),\hat{G}(\mu_q^N)=\text{const.}\prod_{i=1}^{r}(\mu_q^N-w_i^{\epsilon_i}),\tag{3.8}$$

where $\epsilon_i=1$ or -1 is independently chosen for the index i.

The 2^r solutions for $G(\mu_q^N)$ and $\hat{G}(\mu_q^N)$ are similar to the 2^r eigenvalues of the Ising-like form [18, 38]. We thus call the set of 2^r eigenvalues of the diagonal-to-diagonal transfer matrices associated with a regular Bethe state the Ising-like spectrum associated with the regular Bethe state. In fact, in the homogeneous case of $p_1 = \cdots = p_L$, it follows from the Onsager-algebra structure of the SCP model that each eigenvalue is non-degenerate, that is, the multiplicity of the eigenvalue specified by a set of $\{\epsilon_i\}$ is given by one.

For the homogeneous case, the Ising-like spectrum of the diagonal-to-diagonal transfer matrix was shown by applying the functional relations among the transfer matrices [1, 2, 6–8, 41]. There are three types of the functional relations [6–8]: the first relation is based on the fact that the transfer matrix of the SCP model is exactly a Q-operator for the τ_2-model [13, 40], and it gives eigenvalues of the transfer matrix of the τ_2-model. The second relation is interpreted as a T-system [32, 33], which recursively generates the eigenvalues of the transfer matrices in the fusion hierarchy. The third relation leads to the eigenvalues of the product of the diagonal-to-diagonal transfer matrices of the SCP model with a constraint on the spectral parameters. The algebraic Bethe ansatz of the τ_2-model given in the previous section plays the same role as the first functional relation [41]. The algebraic approach formulated in this section plays a similar role as the second and third functional relations.

3.3 Proof of Proposition 3.1

The subsection is devoted to a proof of Proposition 3.1. Our strategy is to derive a recursion relation for the coefficients $c_n^{\tau'\tau}(\{i_\ell\}; \{j_\ell\}; \{k_\ell\})$ in the relation (3.1).

Lemma 3.3 *The Yang-Baxter relation* (2.13) *is equivalent to the following relations*:

$$
\begin{aligned}
\alpha_{\tau'}(z)A(z)T_\tau^{\tau'} + \beta_{\tau'}(z)C(z)T_\tau^{\tau'-1} &= \alpha_\tau(z)T_\tau^{\tau'}A(z) + \gamma_\tau(z)T_{\tau-1}^{\tau'}B(z),\\
\alpha_{\tau'}(z)B(z)T_\tau^{\tau'} + \beta_{\tau'}(z)D(z)T_\tau^{\tau'-1} &= \beta_{\tau+1}(z)T_{\tau+1}^{\tau'}A(z) + \delta_\tau(z)T_\tau^{\tau'}B(z),\\
\gamma_{\tau'+1}(z)A(z)T_\tau^{\tau'+1} + \delta_{\tau'}(z)C(z)T_\tau^{\tau'} &= \alpha_\tau(z)T_\tau^{\tau'}C(z) + \gamma_\tau(z)T_{\tau-1}^{\tau'}D(z),\\
\gamma_{\tau'+1}(z)B(z)T_\tau^{\tau'+1} + \delta_{\tau'}(z)D(z)T_\tau^{\tau'} &= \beta_{\tau+1}(z)T_{\tau+1}^{\tau'}C(z) + \delta_\tau(z)T_\tau^{\tau'}D(z),
\end{aligned}
\tag{3.9}
$$

where

$$
\alpha_\tau(z) = -y_{q_1}y_{q_2}z + \mu_{q_1}\mu_{q_2}\omega^\tau, \qquad \beta_\tau(z) = -z(y_{q_1} - x_{q_2}\mu_{q_1}\mu_{q_2}\omega^\tau),
$$
$$
\gamma_\tau(z) = y_{q_2} - x_{p_1}\mu_{q_1}\mu_{q_2}\omega^\tau, \qquad \delta_\tau(z) = 1 - x_{q_1}x_{q_2}\mu_{q_1}\mu_{q_2}z\omega^{\tau+1}.
$$

Here we have omitted the dependence of the spectral parameters q_1 and q_2 in the coefficients $\alpha_\tau(z), \beta_\tau(z), \gamma_\tau(z), \delta_\tau(z)$ and the operator $T_\tau^{\tau'}$.

Lemma 3.4 *For the operators $T_\tau^{\tau'}$, we have*

$$
\begin{aligned}
&\mu_{\tau'\tau}(z)B(z)T_\tau^{\tau'}\\
&\quad = T_\tau^{\tau'}B(z) + \nu_{\tau+1}(z)T_{\tau+1}^{\tau'}A(z) - \nu_{\tau'}(z)T_\tau^{\tau'-1}D(z) - \nu_{\tau'}(z)\nu_{\tau+1}(z)T_{\tau+1}^{\tau'-1}C(z).
\end{aligned}
\tag{3.10}
$$

Here $\mu_{\tau'\tau}(z)$ and $\nu_\tau(z)$ coincide with those defined in (3.2), *respectively, by setting $q_1 = (x_q, y_q, \mu_q)$ and $q_2 = (y_q, x_q\omega^s, \mu_q^{-1})$.*

Proof From the second and fourth relations in Lemma 3.3, we have

$$
T_\tau^{\tau'}B(z) = \left(\frac{\alpha_{\tau'}(z)}{\delta_\tau(z)} - \frac{\beta_{\tau'}(z)}{\delta_\tau(z)}\frac{\gamma_{\tau'}(z)}{\delta_{\tau'-1}(z)}\right)B(z)T_\tau^{\tau'} - \frac{\beta_{\tau+1}(z)}{\delta_\tau(z)}T_{\tau+1}^{\tau'}A(z)
$$

$$+\frac{\beta_{\tau'}(z)}{\delta_{\tau'-1}(z)}T_{\tau}^{\tau'-1}D(z)+\frac{\beta_{\tau'}(z)}{\delta_{\tau'-1}(z)}\frac{\beta_{\tau+1}(z)}{\delta_{\tau}(z)}T_{\tau+1}^{\tau'-1}C(z).$$

By setting $q_1=(x_q,y_q,\mu_q)$ and $q_2=(y_q,x_q\omega^s,\mu_q^{-1})$, we prove the relation. □

Lemma 3.5 *Let $I=\{i_\ell\}$, $J=\{j_\ell\}$ and $K=\{k_\ell\}$ be such disjoint subsets of the set $\Sigma_n=\{1,2,\ldots,n\}$ that $\sharp I=n_B$, $\sharp J=n_A$, $\sharp K=n_D$ and $n_B+n_A+n_D=n$. The coefficients $c_n(I;J;K)=c_n^{\tau'\tau}(I;J;K)$ in the relation (3.1) satisfy the following recursion relation on n:*

$$\begin{aligned}
&c_n(I;J;K)\\
&\quad=c_{n-1}(I\setminus\{n\};J;K)\frac{1}{\mu_{\tau'-n_D,\tau+n_A}(z_n)}\\
&\qquad+c_{n-1}(I;J\setminus\{n\};K)\frac{\nu_{\tau+n_A}(z_n)}{\mu_{\tau'-n_D,\tau+n_A-1}(z_n)}\prod_{i\in I}\omega f_{ni}\\
&\qquad-\sum_{j\in J}c_{n-1}(I\cup\{j\}\setminus\{n\};J\setminus\{j\};K)\frac{\nu_{\tau+n_A}(z_n)}{\mu_{\tau'-n_D,\tau+n_A-1}(z_n)}\omega\left(\prod_{i\in I\setminus\{n\}}\omega f_{ji}\right)g_{nj}\\
&\qquad-c_{n-1}(I;J;K\setminus\{n\})\frac{\nu_{\tau'-n_D+1}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A}(z_n)}\prod_{i\in I}\omega f_{in}\\
&\qquad-\sum_{k\in K}c_{n-1}(I\cup\{k\}\setminus\{n\};J;K\setminus\{k\})\frac{\nu_{\tau'-n_D+1}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A}(z_n)}\omega\left(\prod_{i\in I\setminus\{n\}}\omega f_{ik}\right)g_{nk}\\
&\qquad-\sum_{k\in K}c_{n-1}(I\cup\{k\};J\setminus\{n\};K\setminus\{k\})\frac{\nu_{\tau'-n_D+1}(z_n)\nu_{\tau+n_A}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A-1}(z_n)}\omega\left(\prod_{i\in I}\omega f_{ni}\omega f_{ik}\right)g_{nk}\\
&\qquad+\sum_{j\in J}c_{n-1}(I\cup\{j\};J\setminus\{j\};K\setminus\{n\})\frac{\nu_{\tau'-n_D+1}(z_n)\nu_{\tau+n_A}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A-1}(z_n)}\omega\left(\prod_{i\in I}\omega f_{ji}\omega f_{in}\right)g_{nj}\\
&\qquad+\sum_{\substack{j\in J\\k\in K}}c_{n-1}(I\cup\{j\}\cup\{k\}\setminus\{n\};J\setminus\{j\};K\setminus\{k\})\\
&\qquad\times\frac{\nu_{\tau'-n_D+1}(z_n)\nu_{\tau+n_A}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A-1}(z_n)}\omega f_{jk}\left(\prod_{i\in I\setminus\{n\}}\omega f_{ji}\omega f_{ik}\right)g_{nj}g_{nk}.
\end{aligned}$$

Here, if the set S_1 is not a subset of S, we set $c_n(S\setminus S_1;\cdot;\cdot)=c_n(\cdot;S\setminus S_1;\cdot)=c_n(\cdot;\cdot;S\setminus S_1)=0$.

Proof We apply the operator $B_n=B(z_n)$ to both sides of (3.1) with $n-1$ in place of n. Let $\tilde I=\{\tilde i_\ell\}$, $\tilde J=\{\tilde j_\ell\}$ and $\tilde K=\{\tilde k_\ell\}$ be such disjoint subsets of the set Σ_{n-1} that $\sharp\tilde I=m_B$, $\sharp\tilde J=m_A$, $\sharp\tilde K=m_D$ and $m_B+m_A+m_D=n-1$. By using the relation (3.10), we have

$$\begin{aligned}
&B_1\cdots B_{n-1}B_nT_\tau^{\tau'}|0\rangle\\
&\quad=\sum_{\tilde I,\tilde J,\tilde K}c_{n-1}^{\tau'\tau}(\tilde I;\tilde J;\tilde K)B_nT_{\tau+m_A}^{\tau'-m_D}B_{\tilde i_1}\cdots B_{\tilde i_{m_B}}A_{\tilde j_1}\cdots A_{\tilde j_{m_A}}D_{\tilde k_1}\cdots D_{\tilde k_{m_D}}|0\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\tilde{I},\tilde{J},\tilde{K}} c_{n-1}^{\tau'\tau}(\tilde{I};\tilde{J};\tilde{K}) \left(\frac{1}{\mu_{\tau'-m_D,\tau+m_A}(z_n)} T_{\tau+m_A}^{\tau'-m_D} B_n + \frac{\nu_{\tau+m_A+1}(z_n)}{\mu_{\tau'-m_D,\tau+m_A}(z_n)} T_{\tau+m_A+1}^{\tau'-m_D} A_n \right. \\
&\quad \left. - \frac{\nu_{\tau'-m_D}(z_n)}{\mu_{\tau'-m_D,\tau+m_A}(z_n)} T_{\tau+m_A}^{\tau'-m_D-1} D_n - \frac{\nu_{\tau'-m_D}(z_n)\nu_{\tau+m_A+1}(z_n)}{\mu_{\tau'-m_D,\tau+m_A}(z_n)} T_{\tau+m_A+1}^{\tau'-m_D-1} C_n \right) \\
&\quad \times B_{\tilde{i}_1} \cdots B_{\tilde{i}_{m_B}} A_{\tilde{j}_1} \cdots A_{\tilde{j}_{m_A}} D_{\tilde{k}_1} \cdots D_{\tilde{k}_{m_D}} |0\rangle.
\end{aligned}
$$

By arranging the operators $A(z)$, $B(z)$, $C(z)$ and $D(z)$ in the order $BADC$ with the relations in Lemma A.1 and by rewriting the terms in the form of the right-hand side of (3.1), we obtain the recursion relation. □

We now prove Proposition 3.1. From the symmetry of the relation (3.1), it is enough to solve the recursion relation in the case $i_1 < \cdots < i_{n_B} < j_1 < \cdots < j_{n_A} < k_1 < \cdots < k_{n_D}$. First we consider the case $n_B = n$, that is, $i_\ell = \ell$ for $\ell = 1, 2, \ldots, n$ and $J = K = \phi$. The recursion relation is reduced to

$$
c_n(I;\phi;\phi) = c_{n-1}(I \setminus \{n\};\phi;\phi) \frac{1}{\mu_{\tau'\tau}(z_n)}.
$$

From the initial condition $c_0(\phi;\phi;\phi) = 1$, the recursion relation is solved as

$$
c_n(I;\phi;\phi) = \prod_{i\in I} \frac{1}{\mu_{\tau'\tau}(z_i)},
$$

which is consistent with the form (3.1). Second we consider the case $n_B + n_A = n$, that is, $n \in \{j_\ell\}$ and $K = \phi$. The recursion relation is reduced to

$$
c_n(I;J;\phi) = c_{n-1}(I;J \setminus \{n\};\phi) \frac{\nu_{\tau+n_A}(z_n)}{\mu_{\tau',\tau+n_A-1}(z_n)} \prod_{i\in I} \omega f_{ni}.
$$

By using the result in the case $n_B = n$, we obtain

$$
c_n(I;J;\phi) = \prod_{i\in I} \frac{1}{\mu_{\tau'\tau}(z_i)} \prod_{p=1}^{n_A} \frac{\nu_{\tau+p}(z_{j_p})}{\mu_{\tau',\tau+p-1}(z_{j_p})} \prod_{\substack{i\in I \\ j\in J}} \omega f_{ji},
$$

which is also consistent with the form (3.1). Third we consider the case $n \in \{k_\ell\}$. The recursion relation is reduced to

$$
\begin{aligned}
&c_n(I;J;K) \\
&\quad = -c_{n-1}(I;J;K \setminus \{n\}) \frac{\nu_{\tau'-n_D+1}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A}(z_n)} \prod_{i\in I} \omega f_{in} \\
&\qquad + \sum_{j\in J} c_{n-1}(I \cup \{j\};J \setminus \{j\};K \setminus \{n\}) \frac{\nu_{\tau'-n_D+1}(z_n)\nu_{\tau+n_A}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A-1}(z_n)} \omega \left(\prod_{i\in I} \omega f_{ji} \omega f_{in} \right) g_{nj}.
\end{aligned}
\tag{3.11}
$$

Note that, in the case, the coefficients $c_{n-1}(\{i'_\ell\};\{j'_\ell\};\{k'_\ell\})$ with general sets $\{i'_\ell\}$, $\{j'_\ell\}$ and $\{k'_\ell\}$, which are not necessarily in the order $i'_1 < \cdots < i'_{n_B} < j'_1 < \cdots < j'_{n_A} < k'_1$

$< \cdots < k'_{n_D}$, appear. Assume that the coefficients $c_{n-1}(I; J; K \setminus \{n\})$ and $c_{n-1}(I \cup \{j\}; J \setminus \{j\}; K \setminus \{n\})$ in (3.11) are given in the form (3.1). Substituting the form of the coefficient $c_{n-1}(\{i_\ell\}; \{j_\ell\}; \{k_\ell\} \setminus \{n\})$ and $k_{n_D} = n$ into the first term of (3.11), we obtain

$$
\begin{aligned}
&-c_{n-1}(\{i_\ell\}; \{j_\ell\}; \{k_\ell\} \setminus \{n\}) \frac{\nu_{\tau'-n_D+1}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A}(z_n)} \prod_{r=1}^{n_B} \omega f_{i_r n} \\
&= -(-)^{n_D-1} \prod_{r=1}^{n_B} \frac{1}{\mu_{\tau'\tau}(z_{i_r})} \prod_{p=1}^{n_A} \frac{\nu_{\tau+p}(z_{j_p})}{\mu_{\tau',\tau+p-1}(z_{j_p})} \prod_{q=1}^{n_D-1} \frac{\nu_{\tau'-q+1}(z_{k_q})}{\mu_{\tau'-q+1,\tau}(z_{k_q})} \frac{\nu_{\tau'-n_D+1}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A}(z_n)} \\
&\quad \times \prod_{\substack{i\in\{i_\ell\}\\ j\in\{j_\ell\}}} \omega f_{ji} \prod_{\substack{i\in\{i_\ell\}\\ k\in\{k_\ell\}\setminus\{n\}}} \omega f_{ik} \prod_{\substack{j\in\{j_\ell\}\\ k\in\{k_\ell\}\setminus\{n\}}} \omega f_{jk} \prod_{r=1}^{n_B} \omega f_{i_r n} \\
&= c_n(\{i_\ell\}; \{j_\ell\}; \{k_\ell\}) \frac{\mu_{\tau'-n_D+1,\tau}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A}(z_n)} \prod_{j\in\{j_\ell\}} \frac{1}{\omega f_{jn}}.
\end{aligned}
\tag{3.12}
$$

In a similar way, substituting the forms of the coefficients $c_{n-1}(\{i_\ell\} \cup \{j_p\}; \{j_\ell\} \setminus \{j_p\}; \{k_\ell\} \setminus \{n\})$ and $k_{n_D} = n$ into the second term of (3.11), we obtain

$$
\begin{aligned}
&\sum_{\substack{p=1\\ j_p\neq n}}^{n_A} c_{n-1}(\{i_\ell\} \cup \{j_p\}; \{j_\ell\} \setminus \{j_p\}; \{k_\ell\} \setminus \{n\}) \\
&\qquad \times \frac{\nu_{\tau'-n_D+1}(z_n)\nu_{\tau+n_A}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A-1}(z_n)} \omega \left(\prod_{r=1}^{n_B} \omega f_{j_p i_r} \omega f_{i_r n} \right) g_{n j_p} \\
&\quad = \sum_{\substack{p=1\\ j_p\neq n}}^{n_A} (-)^{n_D-1} \prod_{r=1}^{n_B} \frac{1}{\mu_{\tau'\tau}(z_{i_r})} \frac{1}{\mu_{\tau'\tau}(z_{j_p})} \prod_{p'=1}^{p-1} \frac{\nu_{\tau+p'}(z_{j_{p'}})}{\mu_{\tau',\tau+p'-1}(z_{j_{p'}})} \\
&\qquad \times \prod_{p'=p+1}^{n_A} \frac{\nu_{\tau+p'-1}(z_{j_{p'}})}{\mu_{\tau',\tau+p'-2}(z_{j_{p'}})} \prod_{q=1}^{n_D-1} \frac{\nu_{\tau'-q+1}(z_{k_q})}{\mu_{\tau'-q+1,\tau}(z_{k_q})} \frac{\nu_{\tau'-n_D+1}(z_n)\nu_{\tau+n_A}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A-1}(z_n)} \omega \\
&\qquad \times \prod_{\substack{i\in\{i_\ell\}\cup\{j_p\}\\ j\in\{j_\ell\}\setminus\{j_p\}}} \omega f_{ji} \prod_{\substack{i\in\{i_\ell\}\cup\{j_p\}\\ k\in\{k_\ell\}\setminus\{n\}}} \omega f_{ik} \prod_{\substack{j\in\{j_\ell\}\setminus\{j_p\}\\ k\in\{k_\ell\}\setminus\{n\}}} \omega f_{jk} \left(\prod_{r=1}^{n_B} \omega f_{j_p i_r} \omega f_{i_r n} \right) g_{n j_p} \\
&\quad = -c_n(\{i_\ell\}; \{j_\ell\}; \{k_\ell\}) \sum_{p=1}^{n_A} \frac{\mu_{\tau',\tau+n_A-1}(z_{j_p})}{\mu_{\tau'\tau}(z_{j_p})} \frac{\mu_{\tau'-n_D+1,\tau}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A-1}(z_n)} \frac{\nu_{\tau+n_A}(z_n)}{\nu_{\tau+n_A}(z_{j_p})} \frac{g_{n j_p}}{f_{j_p n}} \\
&\qquad \times \prod_{j\in\{j_\ell\}\setminus\{j_p\}} \frac{f_{j j_p}}{f_{jn}}.
\end{aligned}
\tag{3.13}
$$

Hence, by combining (3.12) and (3.13), the coefficient $c_n(\{i_\ell\}; \{j_\ell\}; \{k_\ell\})$ is shown to be in the form (3.1) if the following relation holds:

$$\frac{\mu_{\tau'-n_D+1,\tau}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A}(z_n)}\prod_{p=1}^{n_A}\frac{1}{\omega f_{j_p n}}$$

$$-\sum_{p=1}^{n_A}\frac{\mu_{\tau',\tau+n_A-1}(z_{j_p})}{\mu_{\tau'\tau}(z_{j_p})}\frac{\mu_{\tau'-n_D+1,\tau}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A-1}(z_n)}\frac{\nu_{\tau+n_A}(z_n)}{\nu_{\tau+n_A}(z_{j_p})}\frac{g_{nj_p}}{f_{j_p n}}\prod_{j\in\{j_\ell\}\setminus\{j_p\}}\frac{f_{jj_p}}{f_{jn}}=1,$$

which is the identity in Lemma B.3.

4 The $\mathfrak{sl}_2$ Loop Algebra Symmetry of the τ_2-Model and Degenerate Eigenspaces

4.1 Gauge Transformations on the L-Operator

We now introduce another L-operator in order to show the $\mathfrak{sl}_2$ loop algebra symmetry of the τ_2-model. The degenerate eigenspace of the transfer matrix constructed from the new L-operator is identical to the degenerate eigenspace of the τ_2-model which we have introduced in Sect. 2.2.

Let us introduce the L-operator $\tilde{\mathcal{L}}_i(z)\in\mathrm{End}(\mathbb{C}^2\otimes(\mathbb{C}^N)^{\otimes L})$ $(i=1,2,\ldots,L)$ given by

$$\tilde{\mathcal{L}}_i(z)=\begin{pmatrix} q^{-\frac{1}{2}}\big(z(k^{\frac{1}{2}})_i-z^{-1}(k^{-\frac{1}{2}})_i\big) & (q-q^{-1})(f)_i\\ (q-q^{-1})(e)_i & q^{\frac{1}{2}}\big(z(k^{-\frac{1}{2}})_i-z^{-1}(k^{\frac{1}{2}})_i\big)\end{pmatrix}. \tag{4.1}$$

Here q is not a rapidity on the Fermat curve (2.1) but a generic parameter, and $\{(k)_i,(e)_i,(f)_i\}$ is the N-dimensional representation of the quantum algebra $U_q(\mathfrak{sl}_2)$ non-trivially acting only on the ith component of the quantum space $(\mathbb{C}^N)^{\otimes L}$ as

$$kv_\sigma=\varepsilon q^{N-1-2\sigma}v_\sigma,\qquad ev_\sigma=\varepsilon\alpha[N-\sigma]v_{\sigma-1},\qquad fv_\sigma=\alpha^{-1}[\sigma+1]v_{\sigma+1},$$

with $\alpha\neq 0$ and $[n]=\frac{q^n-q^{-n}}{q-q^{-1}}$. We set $\varepsilon=1$ for odd N and $\varepsilon=-1$ for even N. One sees that the L-operator $\tilde{\mathcal{L}}_i(z)$ is nothing but that of an XXZ spin chain with N-state local spins and a twist parameter. The L-operator $\tilde{\mathcal{L}}_i(z)$ satisfies the Yang-Baxter relation (2.5) with the R-matrix of the six-vertex model given by

$$R_{6\mathrm{v}}(z)=\begin{pmatrix} 1-z^2q^2 & 0 & 0 & 0\\ 0 & (1-z^2)q & z(1-q^2) & 0\\ 0 & z(1-q^2) & (1-z^2)q & 0\\ 0 & 0 & 0 & 1-z^2q^2\end{pmatrix}. \tag{4.2}$$

We introduce the monodromy matrix $\tilde{\mathcal{T}}(z;\{p\})\in\mathrm{End}(\mathbb{C}^2\otimes(\mathbb{C}^N)^{\otimes L})$ and the transfer matrix $\tilde{\tau}(z)=\tilde{\tau}(z;\{p\})\in\mathrm{End}((\mathbb{C}^N)^{\otimes L})$ as

$$\begin{aligned}\tilde{\mathcal{T}}(z;\{p\})&=\prod_{i=1}^{L}\tilde{\mathcal{L}}_i(t_{p_i}^{\frac{1}{2}}zq^{\frac{1}{2}})=:\begin{pmatrix}\tilde{A}(z) & \tilde{B}(z)\\ \tilde{C}(z) & \tilde{D}(z)\end{pmatrix},\\ \tilde{\tau}(z;\{p\})&=\mathrm{tr}_{\mathbb{C}^2}\big(\tilde{\mathcal{T}}(z;\{p\})\big).\end{aligned} \tag{4.3}$$

In a way similar to Sect. 2.2, we apply the algebraic Bethe-ansatz method to the transfer matrix $\tilde{\tau}(z;\{p\})$ to obtain Bethe eigenstates. The associated Bethe equations are given by

$$\prod_{n=1}^{L} \frac{t_{p_n} z_i^2 \varepsilon q^N - 1}{t_{p_n} z_i^2 q^2 - \varepsilon q^N} = \prod_{j(\neq i)} \frac{z_i^2 q^2 - z_j^2}{z_i^2 - z_j^2 q^2}. \tag{4.4}$$

The transfer matrix $\tau(z^2;\{p\})$ of the τ_2-model defined in (2.9) is equivalent to the transfer matrix $\tilde{\tau}(z;\{p\})$ at $\varepsilon q^N = 1$. We set $\omega = q^2$ with the primitive Nth root of unity q for odd N and the primitive $2N$th root of unity q for even N, and take $\alpha = x_{p_i}^{\frac{1}{2}} y_{p_i}^{-\frac{1}{2}}$. Then, in terms of the operators Z_i and X_i, the representation of the quantum algebra $U_q(\mathfrak{sl}_2)$ is expressed as

$$(k)_i = q^{-1} Z_i^{-1}, \qquad (e)_i = \frac{x_{p_i}^{\frac{1}{2}} y_{p_i}^{-\frac{1}{2}}}{q - q^{-1}} X_i^{-1}(Z_i^{-\frac{1}{2}} - Z_i^{\frac{1}{2}}), \qquad (f)_i = \frac{x_{p_i}^{-\frac{1}{2}} y_{p_i}^{\frac{1}{2}}}{q - q^{-1}} (Z_i^{\frac{1}{2}} - Z_i^{-\frac{1}{2}}) X_i,$$

by which the L-operator $\tilde{\mathcal{L}}_i(z)$ at $\varepsilon q^N = 1$ takes the form

$$\tilde{\mathcal{L}}_i(z) = \begin{pmatrix} q^{-\frac{1}{2}}\big(-zq^{-\frac{1}{2}} Z_i^{-\frac{1}{2}} + z^{-1} q^{\frac{1}{2}} Z_i^{\frac{1}{2}}\big) & x_{p_i}^{-\frac{1}{2}} y_{p_i}^{\frac{1}{2}} (Z_i^{\frac{1}{2}} - Z_i^{-\frac{1}{2}}) X_i \\ x_{p_i}^{\frac{1}{2}} y_{p_i}^{-\frac{1}{2}} X_i^{-1} (Z_i^{-\frac{1}{2}} - Z_i^{\frac{1}{2}}) & q^{\frac{1}{2}}\big(-zq^{\frac{1}{2}} Z_i^{\frac{1}{2}} + z^{-1} q^{-\frac{1}{2}} Z_i^{-\frac{1}{2}}\big) \end{pmatrix}.$$

The L-operator $\tilde{\mathcal{L}}_i(z)$ is transformed to the L-operator $\mathcal{L}_i(z^2;p_i,\bar{p}_i)$ defined in (2.8) as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & z^{-1} q^{\frac{1}{2}} \end{pmatrix} t_{p_i}^{\frac{1}{2}} z q^{\frac{1}{2}} Z_i^{\frac{1}{2}} \tilde{\mathcal{L}}_i\big(t_{p_i}^{\frac{1}{2}} z q^{\frac{1}{2}}\big) \begin{pmatrix} 1 & 0 \\ 0 & zq^{-\frac{1}{2}} \end{pmatrix} = \mathcal{L}_i(z^2;p_i,\bar{p}_i).$$

Through the gauge transformation, the Yang-Baxter relation with the R-matrix $R_{6\mathrm{v}}(z)$ for the L-operator $\tilde{\mathcal{L}}_i(z)$ is transformed to the Yang-Baxter relation (2.10) with the R-matrix $R(z)$ (2.6). In the case of odd N, the L-operator $\tilde{\mathcal{L}}_i(z)$ satisfies the Yang-Baxter relation (2.7). On the other hand, in the case of even N, the L-operator $\tilde{\mathcal{L}}_i(z)$ does not satisfy the relation (2.7) due to the multiplication by the operator $Z_i^{\frac{1}{2}}$. However, the conserved operators derived from an expansion of the logarithm of the transfer matrix $\tilde{\tau}(z;\{p\})$ commute with the transfer matrix $t(q_1,q_2;\{p\})$ since the operators $Z_i^{\frac{1}{2}}$ are canceled out in the derivation. Furthermore, the product $Z_1^{\frac{1}{2}} \cdots Z_L^{\frac{1}{2}}$, which appears in each entry of the monodromy matrix $\tilde{\mathcal{T}}(z;\{p\})$, acts as the constant q^M on the sector spanned by the vectors $v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_L}$ satisfying $\sigma_1 + \cdots + \sigma_L = M$. As we shall see below, each Bethe eigenstate and its $L(\mathfrak{sl}_2)$-descendant state belong to one of the sectors. Therefore, transfer matrices $\tilde{\tau}(z;\{p\})$ and $\tau(z;\{p\})$ thus share a set of common eigenvectors.

4.2 The $\mathfrak{sl}_2$ Loop Algebra Symmetry

We now show the $\mathfrak{sl}_2$ loop algebra $L(\mathfrak{sl}_2)$ symmetry of the τ_2-model. We first obtain a representation of the quantum affine algebra $U_q'(\mathfrak{sl}_2)$ in a limit of the entries of the monodromy matrix $\tilde{\mathcal{T}}(z;\{p\})$:

$$A := \lim_{z\to\infty} \frac{\tilde{A}(z)}{m(z) q^{-\frac{L}{2}}} = \lim_{z\to 0} \frac{\tilde{D}(z)}{m(z) q^{\frac{L}{2}}} = k^{\frac{1}{2}} \otimes \cdots \otimes k^{\frac{1}{2}},$$

$$B_\pm := \lim_{z^{\pm 1}\to\infty} \frac{\tilde{B}(z)}{m(z)n_\pm(z)} = \sum_{i=1}^{L} q^{\frac{L+1}{2}-i}(t_{p_i}^{\mp\frac{1}{2}} q^{\mp\frac{1}{2}}) \underbrace{k^{\pm\frac{1}{2}} \otimes \cdots \otimes k^{\pm\frac{1}{2}}}_{i-1} \otimes f \otimes \underbrace{k^{\mp\frac{1}{2}} \otimes \cdots \otimes k^{\mp\frac{1}{2}}}_{L-i},$$

$$C_\pm := \lim_{z^{\pm 1}\to\infty} \frac{\tilde{C}(z)}{m(z)n_\pm(z)} = \sum_{i=1}^{L} q^{-\frac{L+1}{2}+i}(t_{p_i}^{\mp\frac{1}{2}} q^{\mp\frac{1}{2}}) \underbrace{k^{\mp\frac{1}{2}} \otimes \cdots \otimes k^{\mp\frac{1}{2}}}_{i-1} \otimes e \otimes \underbrace{k^{\pm\frac{1}{2}} \otimes \cdots \otimes k^{\pm\frac{1}{2}}}_{L-i},$$

where $m(z) = \prod_{i=1}^{L}(t_{p_i}^{\frac{1}{2}} z q^{\frac{1}{2}} - t_{p_i}^{-\frac{1}{2}} z^{-1} q^{-\frac{1}{2}})$ and $n_\pm(z) = \pm z^{\mp 1}(q - q^{-1})$. They indeed give a finite-dimensional representation of $U_q'(\hat{\mathfrak{sl}}_2)$ through the map $\pi^{(L)} : U_q'(\hat{\mathfrak{sl}}_2) \to (\mathbb{C}^N)^{\otimes L}$ defined by

$$\pi^{(L)} : k_{0,1}, e_0, e_1, f_0, f_1 \mapsto A^{\mp 2}, B_+, C_+, C_-, B_-,$$

where $\{k_i, e_i, f_i | i = 0, 1\}$ is a set of the Chevalley generators of $U_q'(\hat{\mathfrak{sl}}_2)$.

Second we show that, in the limit $\varepsilon q^N \to 1$, the representation $\pi^{(L)}$ of the quantum affine algebra $U_q'(\hat{\mathfrak{sl}}_2)$ gives a finite-dimensional representation of a Borel subalgebra of $L(\mathfrak{sl}_2)$. The $\mathfrak{sl}_2$ loop algebra is realized by the Drinfeld generators $\{h_n, x_n^+, x_n^- | n = 0, 1, 2, \ldots\}$ satisfying

$$[h_n, h_m] = 0, \qquad [h_n, x_m^\pm] = \pm 2x_{n+m}^\pm, \qquad [x_n^+, x_m^-] = h_{n+m}.$$

The algebra has two Borel subalgebras $\mathfrak{b}_+$ generated by $\{h_n, x_n^+, x_m^- | n \geqslant 0, m > 0\}$ and $\mathfrak{b}_-$ generated by $\{h_{-n}, x_{-m}^+, x_{-n}^- | n \geqslant 0, m > 0\}$. Define the operators

$$H^{(N)} := \frac{1}{N}\sum_{i=1}^{L} \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes h \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id},$$

$$B_\pm^{(n)} := \lim_{\varepsilon q^N \to 1} \frac{(B_\pm)^n}{[n]!}, \qquad C_\pm^{(n)} := \lim_{\varepsilon q^N \to 1} \frac{(C_\pm)^n}{[n]!} \quad \text{for odd } N,$$

where $h v_\sigma = (N - 1 - 2\sigma) v_\sigma$ and $[n]! = [n][n-1]\cdots[1]$. The operators $B_\pm^{(N)}$ and $C_\pm^{(N)}$ are well-defined in the limit $\varepsilon q^N \to 1$ since both the operators $(B_\pm)^N$ and $(C_\pm)^N$ include the factor $[N]!$ They satisfy the relations

$$\begin{aligned}
&[B_+^{(N)}, B_-^{(N)}] = [C_+^{(N)}, C_-^{(N)}] = 0,\\
&[H^{(N)}, B_\pm^{(N)}] = -2B_\pm^{(N)}, \qquad [H^{(N)}, C_\pm^{(N)}] = 2C_\pm^{(N)},\\
&[B_\pm^{(N)}, [B_\pm^{(N)}, [B_\pm^{(N)}, C_\pm^{(N)}]]] = 0, \qquad [C_\pm^{(N)}, [C_\pm^{(N)}, [C_\pm^{(N)}, B_\pm^{(N)}]]] = 0.
\end{aligned}$$

Here the last two relations are obtained from the limit $\varepsilon q^N \to 1$ of the higher-order q-Serre relations in $U_q'(\hat{\mathfrak{sl}}_2)$ [36]. Then we find that the map $\varphi_+ : \mathfrak{b}_+ \to \mathrm{End}((\mathbb{C}^N)^{\otimes L})$ defined by

$$\varphi_+(h_0) := H^{(N)}, \qquad \varphi_+(x_0^+) := C_+^{(N)}, \qquad \varphi_+(x_1^-) := B_+^{(N)}$$

is extended to a finite-dimensional representation of the Borel subalgebra $\mathfrak{b}_+$ and the map $\varphi_- : \mathfrak{b}_- \to \mathrm{End}((\mathbb{C}^N)^{\otimes L})$ defined by

$$\varphi_-(h_0) := H^{(N)}, \qquad \varphi_-(x_{-1}^+) := C_-^{(N)}, \qquad \varphi_-(x_0^-) := B_-^{(N)}$$

is also extended to that of the Borel subalgebra $\mathfrak{b}_-$.

Proposition 4.1 *The τ_2-model in a sector specified below has the Borel subalgebra symmetry in the following sense: the transfer matrix $\tilde{\tau}(z) = \tilde{\tau}(z; \{p\})$ at $\varepsilon q^N = 1$ satisfies*

$$\left[\tilde{\tau}(1)^{-1}\tilde{\tau}(z), \varphi_+(x)\right] = 0 \quad \textit{for } x \in \mathfrak{b}_+$$

in the sector with $A^2 = q^L$ and

$$\left[\tilde{\tau}(1)^{-1}\tilde{\tau}(z), \varphi_-(x)\right] = 0 \quad \textit{for } x \in \mathfrak{b}_-$$

in the sector with $A^2 = q^{-L}$.

Proof In the limit $\varepsilon q^N \to 1$, we have

$$\begin{aligned}
\tilde{A}(z)B_\pm^{(N)} &= \varepsilon B_\pm^{(N)}\tilde{A}(z) - z^{\pm 1}q^{-\frac{L}{2}}B_\pm^{(N-1)}\tilde{B}(z)A^{\pm 1}, \\
\tilde{D}(z)B_\pm^{(N)} &= \varepsilon B_\pm^{(N)}\tilde{D}(z) + z^{\pm 1}q^{\frac{L}{2}}B_\pm^{(N-1)}\tilde{B}(z)A^{\mp 1}, \\
\tilde{A}(z)C_\pm^{(N)} &= \varepsilon C_\pm^{(N)}\tilde{A}(z) + z^{\pm 1}q^{-\frac{L}{2}}C_\pm^{(N-1)}\tilde{C}(z)A^{\pm 1}, \\
\tilde{D}(z)C_\pm^{(N)} &= \varepsilon C_\pm^{(N)}\tilde{D}(z) - z^{\pm 1}q^{\frac{L}{2}}C_\pm^{(N-1)}\tilde{C}(z)A^{\mp 1}.
\end{aligned}$$

By considering them in the sector with $A^2 = q^{\pm L}$, we prove the proposition. □

Let us consider the condition $A^2 = q^{\pm L}$ in detail. From the relation $A^2 = k \otimes \cdots \otimes k$, we have $A^2 = \varepsilon^L q^{(N-1)L-2M} = q^{-L-2M}$ in the sector spanned by the vectors $v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_L}$ satisfying $\sigma_1 + \cdots + \sigma_L = M$. Then the condition $A^2 = q^L$ means $M + L \equiv 0 \mod N$ and $A^2 = q^{-L}$ means $M \equiv 0 \mod N$. One notices that the reference state $|0\rangle$ belongs to the sector with $A^2 = q^{-L}$.

We now show the $\mathfrak{sl}_2$ loop algebra symmetry of the τ_2-model. It is known that every finite-dimensional irreducible representation of the Borel subalgebra $\mathfrak{b}_\pm$ is extended to that of the $\mathfrak{sl}_2$ loop algebra [14, 22]. Therefore, it follows from Proposition 4.1 that the transfer matrix of the τ_2-model has the $\mathfrak{sl}_2$ loop algebra symmetry.

Third we now show that any given regular Bethe state $|R\rangle$ in the sector with $A^2 = q^{\pm L}$ is a highest weight vector with respect to the representation $\varphi_\pm$ of the Borel subalgebra $\mathfrak{b}_\pm$ and the highest weight representation generated by the Bethe state is irreducible. A vector Ω is called highest weight of the Borel subalgebra $\mathfrak{b}_+$ if it is annihilated by x_n^+ ($n \geqslant 0$) and is diagonalized by h_n ($n \geqslant 0$), and is called highest weight of $\mathfrak{b}_-$ if it is annihilated by x_{-n}^+ ($n > 0$) and is diagonalized by h_{-n} ($n \geqslant 0$). The conditions are equivalent to [21]

$$\begin{aligned}
&x_0^+\Omega = 0, \qquad h_0\Omega = r\Omega, \qquad \frac{(x_0^+)^m}{m!}\frac{(x_1^-)^m}{m!}\Omega = \chi_m^+\Omega \quad (m \in \mathbb{Z}_{>0}) \quad \text{for } \mathfrak{b}_+, \\
&x_{-1}^+\Omega = 0, \qquad h_0\Omega = r\Omega, \qquad \frac{(x_{-1}^+)^m}{m!}\frac{(x_0^-)^m}{m!}\Omega = \chi_m^-\Omega \quad (m \in \mathbb{Z}_{>0}) \quad \text{for } \mathfrak{b}_-,
\end{aligned}$$

where $r \in \mathbb{Z}_{>0}$ and $\chi_m^\pm \in \mathbb{C}$. By using the set $\{\chi_m^\pm\}$ for a highest weight vector of the Borel subalgebra $\mathfrak{b}_\pm$, we define the highest weight polynomial as [23, 24]

$$P_{\mathrm{D}}^\pm(\zeta) = \sum_{m \geqslant 0} \chi_m^\pm (-\zeta)^m.$$

Proposition 4.2 *At $\varepsilon q^N = 1$, every regular Bethe state $|R\rangle$ in the sector with $A^2 = q^{\pm L}$ is a highest weight vector with respect to the representation $\varphi_\pm$ of the Borel subalgebra $\mathfrak{b}_\pm$. The highest weight polynomial is given by*

$$P_{\mathrm{D}}^{\pm}(\xi^N) = \frac{1}{N}\sum_{\tau=0}^{N-1}\left(\prod_{n=1}^{L}\frac{1-t_{p_n}^{\mp N}\xi^N}{1-t_{p_n}^{\mp 1}\xi q^{\pm 2\tau}}\right)\frac{1}{F_\pm(\xi q^{\pm 2\tau})F_\pm(\xi q^{\pm 2(\tau+1)})}, \tag{4.5}$$

where $F_\pm(\xi) = \prod_{i=1}^{R}(1-\xi z_i^{\pm 2})$ and $\{z_i\}$ is a regular solution of the Bethe equations (4.4).

We shall give a proof of Proposition 4.2 in Appendix C.

Here we can directly show that every highest weight vector of the Borel subalgebra becomes a highest weight vector of the $\mathfrak{sl}_2$ loop algebra in a finite-dimensional highest weight representation (see, Appendix A of [25]).

Let us discuss a physical consequence of generic inhomogeneous parameters. For a given regular Bethe state, the zeros of polynomial $P_{\mathrm{D}}^{\pm}(\zeta)$ (4.5) should be distinct, if inhomogeneous parameters $\{p_n\}$ on the Fermat curve (2.1) are given by generic values. If they are distinct, it therefore follows that the highest weight representation generated by the regular Bethe state is irreducible and the polynomial $P_{\mathrm{D}}^{\pm}(\zeta)$ is identified with the Drinfeld polynomial [15, 23, 24, 27]. Assuming that the zeros of the Drinfeld polynomial are distinct, we express the distinct zeros $P_{\mathrm{D}}^{\pm}(\zeta)$ by ζ_i, $(i = 1, 2, \ldots, r = \deg P_{\mathrm{D}}^{\pm}(\zeta))$. Then, the representation is isomorphic to the tensor product of two-dimensional evaluation representations, $V_1(\zeta_1)\otimes\cdots\otimes V_1(\zeta_r)$, and the τ_2-model in the sector $A^2 = q^{\pm L}$ has the 2^r-dimensional degenerate eigenspace associated with the regular Bethe state.

4.3 Complete N-Strings and Degenerate Eigenvectors of the $\mathfrak{sl}_2$ Loop Algebra

In Propositions 4.1 and 4.2 of Sect. 4.2, it has been shown in the sector that the τ_2-model has the $\mathfrak{sl}_2$ loop algebra $L(\mathfrak{sl}_2)$ symmetry and also that every regular Bethe state $|R\rangle$ is a highest weight vector of $L(\mathfrak{sl}_2)$. Therefore, the degenerate eigenspace of the τ_2-model associated with the regular Bethe state $|R\rangle$ is given by the highest weight representation generated by $|R\rangle$ through generators of $L(\mathfrak{sl}_2)$.

Let us define a complete N-string by the set $\{e^{\Lambda}\omega^{-l}|l = 1, 2, \ldots, N\}$, where we call Λ the center of the string [29]. By adding m complete N-strings $\{e^{\Lambda_j}\omega^{-l}|l = 1, 2, \ldots, N, j = 1, 2, \ldots, m\}$ to a regular solution $\{z_i|i = 1, 2, \ldots, R\}$ of the Bethe equations (2.11) and taking the limit $\Lambda_j \to \pm\infty$, we obtain a formal solution $\{z_i\}\cup\{e^{\Lambda_j}\omega^{-l}\}$ of the Bethe equations (2.11) with $M = R + mN$. We call it a non-regular solution. It is clear that the transfer-matrix eigenvalue (2.12) for a non-regular solution $\{z_i\}\cup\{e^{\Lambda_j}\omega^{-l}\}$ is the same as that of the original regular solution $\{z_i\}$.

We now discuss that the SCP transfer matrix $t(q, \bar{q}(s); \{p\})$ with $q = (x_q, y_q, \mu_q)$ and $\bar{q}(s) = (y_q, x_q\omega^s, \mu_q^{-1})$ should have degenerate eigenspaces. We observe that the eigenvalue (3.4) of the SCP transfer matrix $t(q, \bar{q}(s); \{p\})$ with a non-regular solution $\{z_i\}\cup\{e^{\Lambda_j}\omega^{-l}\}$ is the same as that with the original regular solution $\{z_i\}$. As a consequence, the degenerate eigenspace of the transfer matrix $\tau(z; \{p\})$, which contains a regular Bethe state and non-regular Bethe states, corresponds to a degenerate eigenspace of the transfer matrix $t(q, \bar{q}(s); \{p\})$.

Non-regular Bethe eigenstates with complete N-strings may vanish as we shall see in Sect. 4.2. However, there are several approaches to obtain non-zero eigenstates corresponding to non-regular solutions such as complete N-strings [19, 20, 25, 26, 28]. Thus, from the observation that the eigenvalue (3.4) does not depend on complete N-strings, we suggest that the SCP transfer matrix $t(q,\bar{q}(s);\{p\})$ is also degenerate in the $L(\mathfrak{sl}_2)$-degenerate eigenspace of the τ_2-model generated by a regular Bethe state.

We thus propose a conjecture that the $\mathfrak{sl}_2$ loop algebra symmetry of the τ_2-model gives a degenerate eigenspace of the SCP transfer matrix $t(q,\bar{q}(s);\{p\})$ in the sector. Intuitively, in terms of complete N-strings, we may interpret that every $L(\mathfrak{sl}_2)$-descendant state of a given regular Bethe state should be expressed as some linear combination of such non-regular Bethe states consisting of complete N-strings. Furthermore, the Drinfeld polynomial $P_{\mathrm{D}}^{\pm}(\zeta)$ (4.5) is identical to the SCP polynomial $P_{\mathrm{SCP}}(\zeta)$ (3.7) associated with the regular Bethe state. Thus, the $L(\mathfrak{sl}_2)$-degenerate eigenspace of the τ_2-model should have exactly the same dimensions as the invariant subspace associated with the Ising-like spectrum (3.6) characterized by the SCP polynomial $P_{\mathrm{SCP}}(\zeta)$.

4.4 The $\mathfrak{sl}_2$ Loop Algebra Degeneracy and the Ising-Like Spectrum

We now discuss an important consequence of the commutativity of the SCP transfer matrix $t(q,\bar{q}(s);\{p\})$ with the transfer matrix $\tau(z,\{p\})$ of the τ_2-model. Here we note that basis vectors diagonalizing commuting transfer matrices do not depend on the spectral parameters.

We define the completeness of the Bethe ansatz of the τ_2-model at the superintegrable point by the following conjecture:

Conjecture 4.3 All regular Bethe states in the sector with $A^2=q^{\pm L}$ and their descendants with respect to the $\mathfrak{sl}_2$ loop algebra give the complete set of the Hilbert space in the sector on which transfer matrix $\tau(z,\{p\})$ of the τ_2-model acts. Here we recall $\varepsilon q^N=1$.

For generic values of spectral parameter z, regular Bethe states in the sector are non-degenerate with respect to the eigenvalue of transfer matrix $\tau(z,\{p\})$. The degeneracy in the eigenspectrum of transfer matrix $\tau(z,\{p\})$ should be given only by the $\mathfrak{sl}_2$ loop algebra symmetry. Similarly, for generic spectral parameters, regular Bethe states are non-degenerate with respect to the eigenvalue of the SCP transfer matrix $t(q,\bar{q}(s);\{p\})$. The eigenvalue of $t(q,\bar{q}(s);\{p\})$ is also generic with respect to the spectral parameters, as shown in (3.4).

Thus, if Conjecture 4.3 is valid, i.e. the completeness of the Bethe ansatz for the τ_2-model is valid, we have the following corollary:

Corollary 4.4 *In the sector with $A^2=q^{\pm L}$, the SCP transfer matrix $t(q_1,q_2;\{p\})$ is block-diagonalized with respect to the $L(\mathfrak{sl}_2)$-degenerate eigenspaces of the τ_2-model associated with the regular Bethe states. Here we recall $\varepsilon q^N=1$.*

Assuming the arguments for deriving the formula of eigenvalues of the diagonal-to-diagonal transfer matrices $T_{\mathrm{D}}(x_q,y_q)$ and $\hat{T}_{\mathrm{D}}(x_q,y_q)$, we have the following conjecture:

Conjecture 4.5 In the $L(\mathfrak{sl}_2)$-degenerate eigenspace of the τ_2-model associated with a regular Bethe state $|R\rangle$, the diagonal-to-diagonal transfer matrices $T_{\mathrm{D}}(x_q,y_q)$ and $\hat{T}_{\mathrm{D}}(x_q,y_q)$ of the SCP model have the Ising-like spectrum (3.6) associated with the regular Bethe state $|R\rangle$.

Let us consider some examples of the invariant subspace of the Ising-like spectrum associated with a regular Bethe state $|R\rangle$. If the degree of the Drinfeld polynomial $P_{\mathrm{D}}^{\pm}(\zeta)$ is zero, then $|R\rangle$ is an eigenvector of both of the two diagonal-to-diagonal transfer matrices $T_{\mathrm{D}}(x_q, y_q)$ and $\hat{T}_{\mathrm{D}}(x_q, y_q)$. The Bethe state should generate a singlet of the $\mathfrak{sl}_2$ loop algebra, i.e. a one-dimensional highest weight representation. One notices that, if $R = L(N-1)/2$, the degree of the Drinfeld polynomial $P_{\mathrm{D}}^{\pm}(\zeta)$ is zero.

However, if the degree of the Drinfeld polynomial is nonzero and given by r, the SCP transfer matrix $t(q_1, q_2; \{p\})$ should be block-diagonalized at least with respect to the $L(\mathfrak{sl}_2)$-degenerate eigenspace of the τ_2-model associated with $|R\rangle$. Furthermore, the SCP transfer matrix $t(q, \bar{q}(s); \{p\})$ should be degenerate in the 2^r-dimensional $L(\mathfrak{sl}_2)$-degenerate eigenspace of the τ_2-model associated with $|R\rangle$, as it was conjectured in Sect. 4.3.

4.5 $N = 2$ case

We verify in the case of $N = 2$ with a set of homogeneous parameters that the Hamiltonian of the SCP model has the Ising-like spectrum in the $L(\mathfrak{sl}_2)$-degenerate eigenspace of the τ_2-model. In the case, the SCP model is the two-dimensional Ising model. The Hamiltonians of the SCP model and the τ_2-model are given in the forms

$$H_{\mathrm{SCP}} = \sum_{i=1}^{L} \sigma_i^z + \lambda \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x, \qquad H_{\tau_2} = \sum_{i=1}^{L} (\sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^x),$$

where σ^x, σ^y and σ^z are Pauli's matrices. In terms of Jordan-Wigner's fermion operators:

$$c_i = \sigma_i^+ \prod_{j=1}^{i-1} \sigma_j^z, \qquad \tilde{c}_k = \frac{1}{L} \sum_{i=1}^{L} \mathrm{e}^{-\sqrt{-1}(ki+\frac{\pi}{4})} c_i,$$

the Hamiltonian H_{τ_2} in the sector with $S^z := \frac{1}{2}\sum_{i=1}^{L} \sigma_i^z \equiv 0 \mod 2$ is written as

$$H_{\tau_2} = \sum_{k \in K} \sin(k) \tilde{c}_k^\dagger \tilde{c}_k,$$

where $K = \{\frac{\pi}{L}, \frac{3\pi}{L}, \ldots, \frac{(L-1)\pi}{L}\}$. For even L, the $\mathfrak{sl}_2$ loop algebra symmetry describing a degenerate eigenspace of the τ_2-model is given by

$$h_n = \sum_{k \in K} \cot^{2n}\left(\frac{k}{2}\right)(H)_k,$$

$$x_n^+ = \sum_{k \in K} \cot^{2n+1}\left(\frac{k}{2}\right)(E)_k, \qquad x_n^- = \sum_{k \in K} \cot^{2n-1}\left(\frac{k}{2}\right)(F)_k,$$

where $\{(H)_k, (E)_k, (F)_k\}$ is a two-dimensional representation of the $\mathfrak{sl}_2$ algebra given by

$$(H)_k = 1 - \tilde{c}_k^\dagger \tilde{c}_k - \tilde{c}_{-k}^\dagger \tilde{c}_{-k}, \qquad (E)_k = \tilde{c}_{-k} \tilde{c}_k, \qquad (F)_k = \tilde{c}_k^\dagger \tilde{c}_{-k}^\dagger.$$

Here we should remark that for the XX model under the periodic boundary conditions the Chevalley generators of the $\mathfrak{sl}_2$ loop algebra symmetry were constructed in terms of the free fermion operators [26].

The reference state $|0\rangle$, which is a highest weight vector, i.e. $x_n^+|0\rangle = 0$ and $h_n|0\rangle = \sum_k \cot^{2n}(k/2)|0\rangle$, generates a $2^{L/2}$-dimensional irreducible representation corresponding to a degenerate eigenspace of the Hamiltonian H_{τ_2}. On the other hand, in the sector, the Hamiltonian H_{SCP} is expressed as

$$H_{\mathrm{SCP}} = 2\sum_{k\in K}(H)_k - 2\lambda\sum_{k\in K}\big(\cos(k)(H)_k + \sin(k)\big((E)_k + (F)_k\big)\big).$$

It is clear that the Hamiltonian H_{SCP} acts on the $2^{L/2}$-dimensional irreducible representation space. The $2^{L/2}$ eigenvalues of H_{SCP} and the corresponding eigenstates are given by

$$E(K_+;K_-) = 2\sum_{k\in K_+}\sqrt{1-2\lambda\cos(k)+\lambda^2} - 2\sum_{k\in K_-}\sqrt{1-2\lambda\cos(k)+\lambda^2},$$

$$|K_+;K_-\rangle = \prod_{k\in K_+}(\cos\theta_k + \sin\theta_k(F)_k)\prod_{k\in K_-}(\sin\theta_k - \cos\theta_k(F)_k)|0\rangle,$$

where K_+ and K_- are such disjoint subsets of K that $K = K_+\cup K_-$ and $\tan(2\theta_k) = \frac{\lambda\sin(k)}{\lambda\cos(k)-1}$.

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Appendix A: Relations among the Operators in (2.9)

Lemma A.1 *Let A_i, B_i, C_i and D_i denote $A(z_i), B(z_i), C(z_i)$ and $D(z_i)$, respectively. We have*

$$\begin{aligned}
&A_0 B_{i_1}\cdots B_{i_n}\\
&\quad = \omega^n\left(\left(\prod_{p=1}^{n} f_{0i_p}\right)B_{i_1}\cdots B_{i_n}A_0 - \sum_{p=1}^{n}\left(\prod_{q(\neq p)} f_{i_p i_q}\right) g_{0i_p}B_0 B_{i_1}\cdots\overset{i_p}{\check{}}\cdots B_{i_n}A_{i_p}\right),\\
&D_0 B_{i_1}\cdots B_{i_n}\\
&\quad = \omega^n\left(\left(\prod_{p=1}^{n} f_{i_p 0}\right)B_{i_1}\cdots B_{i_n}D_0 + \sum_{p=1}^{n}\left(\prod_{q(\neq p)} f_{i_q i_p}\right) g_{0i_p}B_0 B_{i_1}\cdots\overset{i_p}{\check{}}\cdots B_{i_n}D_{i_p}\right),\\
&C_0 A_{i_1}\cdots A_{i_n}\\
&\quad = \omega^n\left(\left(\prod_{p=1}^{n} f_{i_p 0}\right)A_{i_1}\cdots A_{i_n}C_0 + \sum_{p=1}^{n}\left(\prod_{q(\neq p)} f_{i_q i_p}\right) g_{0i_p}A_0 A_{i_1}\cdots\overset{i_p}{\check{}}\cdots A_{i_n}C_{i_p}\right),\\
&C_0 D_{i_1}\cdots D_{i_n}\\
&\quad = \omega^n\left(\left(\prod_{p=1}^{n} f_{0i_p}\right)D_{i_1}\cdots D_{i_n}C_0 - \sum_{p=1}^{n}\left(\prod_{q(\neq p)} f_{i_p i_q}\right) g_{0i_p}D_0 D_{i_1}\cdots\overset{i_p}{\check{}}\cdots D_{i_n}C_{i_p}\right),
\end{aligned}$$

$$
\begin{aligned}
&D_0 A_{i_1} \cdots A_{i_n} \\
&\quad = A_{i_1} \cdots A_{i_n} D_0 \\
&\qquad + \omega^n \sum_{p=1}^{n} \left(\prod_{q(\neq p)} f_{i_q i_p} \right) \left(g_{0 i_p} B_0 A_{i_1} \cdots \overset{i_p}{\check{}} \cdots A_{i_n} C_{i_p} + g_{i_p 0} B_{i_p} A_{i_1} \cdots \overset{i_p}{\check{}} \cdots A_{i_n} C_0 \right), \\
&C_0 B_{i_1} \cdots B_{i_n} \\
&\quad = \omega^n B_{i_1} \cdots B_{i_n} C_0 \\
&\qquad + \omega^{2n-1} \Bigg(\sum_{p=1}^{n} g_{0 i_p} B_{i_1} \cdots \overset{i_p}{\check{}} \cdots B_{i_n} \\
&\qquad \times \left(\left(\prod_{q(\neq p)} f_{0 i_q} f_{i_q i_p} \right) A_0 D_{i_p} - \left(\prod_{q(\neq p)} f_{i_p i_q} f_{i_q 0} \right) A_{i_p} D_0 \right) \\
&\qquad - \sum_{p \neq q} g_{0 i_p} g_{0 i_q} B_0 B_{i_1} \cdots \overset{i_p}{\check{}} \cdots \overset{i_q}{\check{}} \cdots B_{i_n} f_{i_p i_q} \left(\prod_{r(\neq p,q)} f_{i_p i_r} f_{i_r i_q} \right) A_{i_p} D_{i_q} \Bigg),
\end{aligned}
\tag{A.1}
$$

where $f_{ij} = f(z_i/z_j)$ *and* $g_{ij} = g(z_i/z_j)$ *with*

$$
f(z) = \frac{z-\omega}{(z-1)\omega}, \qquad g(z) = \frac{1-\omega}{(z-1)\omega}.
$$

Proof The relations with $n = 1$ are equivalent to the Yang-Baxter relation (2.10). For $n \geqslant 2$, we employ induction on n with the identity in Lemma B.2. □

Appendix B: Identities

We collect here several useful identities.

Lemma B.1 *Let S be a subset of $\Sigma_R = \{1, 2, \ldots, R\}$. We then have*

$$
\prod_{i \in S} \left(a_i \prod_{j \in \Sigma_R \setminus S} f_{ij} \right) = \prod_{i \in S} \left(d_i \prod_{j \in \Sigma_R \setminus S} f_{ji} \right).
$$

The following three identities of rational functions are proved by verifying that all the residues in the left-hand side are zero.

Lemma B.2

$$
\left(\left(\prod_{i=1}^{n} f_{ik} \right) - \left(\prod_{i=1}^{n} f_{il} \right) \right) g_{kl} + \sum_{i=1}^{n} \left(\prod_{j(\neq i)} f_{ij} \right) g_{ki} g_{il} = 0.
$$

Let $\{i_\ell\}$, $\{j_\ell\}$ and $\{k_\ell\}$ be such disjoint subsets of the set $\Sigma_n = \{1, 2, \ldots, n\}$ that $\sharp\{i_\ell\} = n_B$, $\sharp\{j_\ell\} = n_A$, $\sharp\{k_\ell\} = n_D$ and $n_B + n_A + n_D = n$. We have the following identities:

Lemma B.3

$$\omega^{n_A}\prod_{p=1}^{n_A} f_{j_p,n}-\frac{\mu_{\tau'-n_D+1,\tau}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A}(z_n)}$$
$$+\sum_{p=1}^{n_A}\frac{\mu_{\tau',\tau+n_A-1}(z_{j_p})}{\mu_{\tau'\tau}(z_{j_p})}\frac{\mu_{\tau'-n_D+1,\tau}(z_n)}{\mu_{\tau'-n_D+1,\tau+n_A-1}(z_n)}\frac{\nu_{\tau+n_A}(z_n)}{\nu_{\tau+n_A}(z_{j_p})}\omega^{n_A}g_{n,j_p}\prod_{\substack{r=1\\ r(\neq p)}}^{n_A} f_{j_r,j_p}=0$$

or, explicitly,

$$\prod_{p=1}^{n_A}\frac{z_{n_B+p}-z_n\omega}{z_{n_B+p}-z_n}-\frac{tz_n\omega^{\tau+n_A+1}-1}{tz_n\omega^{\tau+1}-1}$$
$$+\sum_{p=1}^{n_A}\frac{tz_{n_B+p}\omega^{\tau+1}-1}{tz_n\omega^{\tau+1}-1}\frac{z_n(1-\omega)}{z_n-z_{n_B+p}}\prod_{\substack{r=1\\ r(\neq p)}}^{n_A}\frac{z_{n_B+r}-z_{n_B+p}\omega}{z_{n_B+r}-z_{n_B+p}}=0.$$

Lemma B.4

$$\sum_{\substack{\{j_\ell\},\{k_\ell\}\\ n_B+n_A+n_D=n}}\prod_{p=1}^{n_A}\frac{\mu_{\tau+n_D,\tau+n_D}(z_{j_p})}{\mu_{\tau+n_D,\tau+n_D+p-1}(z_{j_p})}\nu_{\tau+n_D+p}(z_{j_p})$$
$$\times\prod_{q=1}^{n_D}\frac{-\mu_{\tau+n_D,\tau+n_D}(z_{k_q})}{\mu_{\tau+n_D-q+1,\tau+n_D}(z_{k_q})}\nu_{\tau+n_D-q+1}(z_{k_q})$$
$$\times\prod_{\substack{i\in\{i_\ell\}\\ j\in\{j_\ell\}}}\omega f_{j_p i_r}\prod_{\substack{i\in\{i_\ell\}\\ k\in\{k_\ell\}}}\omega f_{k_q i_r}\prod_{\substack{j\in\{j_\ell\}\\ k\in\{k_\ell\}}}\omega f_{k_q j_p}=0$$

or, explicitly,

$$\sum_{\substack{\{j_\ell\},\{k_\ell\}\\ n_A+n_D=n-n_B}}(-)^{n_D}\prod_{p=1}^{n_A}\frac{1}{tz_{j_p}\omega^{\tau+n_D+1}-1}\prod_{q=1}^{n_D}\frac{\omega^{q-1}}{tz_{k_q}\omega^{\tau+n_D}-1}\prod_{p=1}^{n_A}\prod_{q=1}^{n_D}\frac{z_{k_q}-z_{j_p}\omega}{z_{k_q}-z_{j_p}}=0.$$

Appendix C: Proof of Proposition 4.2

We give a proof of Proposition 4.2. The detailed proof for the case of the XXZ-Heisenberg spin chain at roots of unity is presented in [21]. Here we show only some different points from it.

For simplicity, we consider the representation φ_+ of the Borel subalgebra $\mathfrak{b}_+$. Let $\tilde{A}_i = \tilde{A}(z_i)$, $\tilde{B}_i = \tilde{B}(z_i)$, $\tilde{C}_i = \tilde{C}(z_i)$ and $\tilde{D}_i = \tilde{D}(z_i)$ for $i \in \Sigma_M = \{1, 2, \ldots, M\}$. One of the relations in Lemma A.1 is rewritten as follows:

$$\begin{aligned}
&\tilde{C}_0\tilde{B}_1\cdots\tilde{B}_M \\
&\quad = \tilde{B}_1\cdots\tilde{B}_M\tilde{C}_0 + \sum_{i=1}^{M}\tilde{g}_{0i}\tilde{B}_1\cdots\overset{i}{\check{}}\cdots\tilde{B}_M\left(\left(\prod_{j(\neq i)}\tilde{f}_{0j}\tilde{f}_{ji}\right)\tilde{A}_0\tilde{D}_i - \left(\prod_{j(\neq i)}\tilde{f}_{j0}\tilde{f}_{ij}\right)\tilde{A}_i\tilde{D}_0\right) \\
&\qquad - \sum_{i\neq j}\tilde{g}_{0i}\tilde{g}_{0j}\tilde{B}_0\tilde{B}_1\cdots\overset{i}{\check{}}\cdots\overset{j}{\check{}}\cdots\tilde{B}_M\tilde{f}_{ij}\left(\prod_{l(\neq i,j)}\tilde{f}_{il}\tilde{f}_{lj}\right)\tilde{A}_i\tilde{D}_j,
\end{aligned} \tag{C.1}$$

where

$$\tilde{f}_{ij} = \tilde{f}(z_i/z_j) = \frac{z_i^2q^{-1} - z_j^2q}{z_i^2 - z_j^2}, \qquad \tilde{g}_{ij} = \tilde{g}(z_i/z_j) = \frac{z_iz_j(q^{-1}-q)}{z_i^2 - z_j^2}.$$

Lemma C.1 *Let $S_n = \{i_1, i_2, \ldots, i_n\}$ be a subset of the set Σ_M. We have*

$$(C_+)^n\left(\prod_{l\in\Sigma_M}\tilde{B}_l\right)|0\rangle = \Delta(S_n;\Sigma_M)\sum_{S_n\subset\Sigma_M}\left(\prod_{l\in\Sigma_M\setminus S_n}\tilde{B}_l\right)|0\rangle \tag{C.2}$$

with the coefficient $\Delta(S_n;\Sigma_M)$ given by

$$\Delta(S_n;\Sigma_M) = \left(\prod_{i\in S_n}z_i\right)\sum_{P\in\mathfrak{S}_n}\sum_{l=0}^{n}(-)^l\begin{bmatrix}n\\l\end{bmatrix}q^{\frac{n(n-1)}{2}-(n-1)l}\prod_{1\leqslant j\leqslant n-l}\alpha_{i_{Pj}}^{\Sigma_M\setminus S_n}\prod_{n-l<j\leqslant n}\bar{\alpha}_{i_{Pj}}^{\Sigma_M\setminus S_n}\sum_{1\leqslant r<s\leqslant n}\tilde{f}_{i_{Pr},i_{Ps}}.$$

Here $\mathfrak{S}_n$ is the symmetric group of order n acting on the set $\{1,2,\ldots,n\}$ and

$$\alpha_i^S = \alpha^S(z_i) := q^{-\frac{N-1}{2}L}\prod_{n=1}^{L}(t_{p_n}^{\frac{1}{2}}z_i\epsilon^{\frac{1}{2}}q^{\frac{N}{2}} - t_{p_n}^{-\frac{1}{2}}z_i^{-1}\epsilon^{-\frac{1}{2}}q^{-\frac{N}{2}})\prod_{j\in S}q\tilde{f}_{ij},$$

$$\bar{\alpha}_i^S = \bar{\alpha}^S(z_i) := q^{\frac{N-1}{2}L}\prod_{n=1}^{L}(t_{p_n}^{\frac{1}{2}}z_i\epsilon^{-\frac{1}{2}}q^{-\frac{N}{2}+1} - t_{p_n}^{-\frac{1}{2}}z_i^{-1}\epsilon^{\frac{1}{2}}q^{\frac{N}{2}-1})\prod_{j\in S}q^{-1}\tilde{f}_{ji}.$$

Proof The case $n=1$ is obtained in the limit $z_0\to\infty$ of the relation (C.1) divided by the factor $m(z_0)n_+(z_0)$. For general n, we use induction on n. □

We consider a diagonal condition $x_0^+x_1^-\Omega = \chi_1^+\Omega$ for the regular Bethe state $|R\rangle$ in the sector with $A^2 = q^L$. Set $M = R+N$ and $n = N$ in (C.2). Let $\{z_i|1\leqslant i\leqslant R\}$ be a regular solution of the Bethe equations (4.4) and put $\{z_{R+l}^2 = \epsilon_l^{-1} := \mathrm{e}^{\Lambda}q^{-2l}|1\leqslant l\leqslant N\}$. We assume that the solution $\{z_i|1\leqslant i\leqslant R\}$ is also regular in the limit $\varepsilon q^N\to 1$. Then we have

$$(C_+)^N\left(\prod_{l\in Z_N}\tilde{B}_l\right)\left(\prod_{i\in\Sigma_R}\tilde{B}_i\right)|0\rangle = \Delta(S_N;\Sigma_{R+N})\sum_{S_n\subset\Sigma_{R+N}}\left(\prod_{l\in\Sigma_{R+N}\setminus S_N}\tilde{B}_l\right)|0\rangle,$$

where $Z_N := \{R+1,\ldots,R+N\}$. We investigate the diagonal term $S_N = Z_N$ in the limit $\Lambda\to-\infty$ after dividing both sides of the equation by the factor $\prod_{l\in Z_N}m(z_{R+l})n_+(z_{R+l})$.

Lemma C.2 *We have*

$$
\begin{aligned}
&\hat{\Delta}(Z_N;\Sigma_{R+N})\\
&\quad:=\left(\prod_{l\in Z_N}\frac{1}{m(z_l)n_+(z_l)}\right)\Delta(Z_N;\Sigma_{R+N})\\
&\quad=\epsilon_0^{-N}\frac{[N]!}{q^{\frac{N(N+1)}{2}+N}(q-q^{-1})^N}\Phi(\epsilon_0)\sum_{l=0}^{N}(-)^l\begin{bmatrix}N\\ l\end{bmatrix}q^{-(N-1)l}\frac{\prod_{j=1}^{N-1}\phi_+(\epsilon_{j+l}\varepsilon q^{-N})}{F_+(\epsilon_l)F_+(\epsilon_{l+1})},
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_+(\xi)&=\prod_{n=1}^{L}(1-t_n^{-1}\xi),\qquad F_+(\xi)=\prod_{i=1}^{R}(1-z_i^2\xi),\\
\Phi(\epsilon_0)&:=\frac{\phi_+(\epsilon_0\varepsilon q^N)F_+(\epsilon_0)F_+(\epsilon_{N+1})}{\prod_{j=1}^{N}\phi_+(\epsilon_j q^{-1})}.
\end{aligned}
$$

Proof From $S_N=\{i_1,\ldots,i_N\}=\{R+1,\ldots,R+N\}$, we have

$$
\begin{aligned}
&\prod_{l\in Z_N}\frac{1}{m(z_l)}\sum_{P\in\mathfrak{S}_N}\prod_{1\leqslant j\leqslant N-l}\alpha_{R+Pj}^{\Sigma_R}\prod_{N-l<j\leqslant N}\bar{\alpha}_{R+Pj}^{\Sigma_R}\prod_{1\leqslant r<s\leqslant N}\tilde{f}_{R+Pr,R+Ps}\\
&\quad=\prod_{l<j\leqslant N}\left(\frac{\phi_+(\epsilon_j\varepsilon q^{-N})}{\phi_+(\epsilon_j q^{-1})}\frac{F_+(\epsilon_{j+1})}{F_+(\epsilon_j)}\right)\prod_{1\leqslant j\leqslant l}\left(\frac{\phi_+(\epsilon_j\varepsilon q^{N-2})}{\phi_+(\epsilon_j q^{-1})}\frac{F_+(\epsilon_{j-1})}{F_+(\epsilon_j)}\right)[N]!\\
&\quad=\frac{\phi_+(\epsilon_0\varepsilon q^N)F_+(\epsilon_0)F_+(\epsilon_{N+1})}{\prod_{j=1}^{N}\phi_+(\epsilon_j q^{-1})}\frac{\prod_{j=1}^{N-1}\phi_+(\epsilon_{j+l}\varepsilon q^{-N})}{F_+(\epsilon_l)F_+(\epsilon_{l+1})}[N]!
\end{aligned}
$$

Here we have used the fact that $\prod_{1\leqslant r<s\leqslant N}\tilde{f}_{R+Pr,R+Ps}=0$ unless P is the longest element in the symmetric group $\mathfrak{S}_N$. □

We define $\tilde{\chi}_m^+$ by the following series expansion:

$$
\frac{\prod_{j=1}^{N-1}\phi_+(\xi\varepsilon q^{2j-N-1})}{F_+(\xi q)F_+(\xi q^{-1})}=\sum_{m\geqslant 0}\tilde{\chi}_m^+(-\xi)^m. \tag{C.3}
$$

Lemma C.3 *In the limit* $\Lambda\to\infty$, *that is*, $\epsilon_0\to 0$, *we have*

$$
\hat{\Delta}(Z_N;\Sigma_{R+N})=\tilde{\chi}_N^+([N]!)^2+O(\epsilon_0).
$$

Proof Put $\xi = \epsilon_0 q^{2l+1}$ in the definition of χ_N^+ (C.3). Then

$$\begin{aligned}
&\sum_{l=0}^{N}(-)^l \begin{bmatrix} N \\ l \end{bmatrix} q^{-(N-1)l} \frac{\prod_{j=1}^{N-1}\phi_+(\epsilon_0\varepsilon q^{2j+2l-N})}{F_+(\epsilon_0 q^{2l})F_+(\epsilon_0 q^{2l+2})} \\
&\quad = \sum_{l=0}^{N}(-)^l \begin{bmatrix} N \\ l \end{bmatrix} \sum_{m=0}^{\infty} \tilde{\chi}_m^+ (-\epsilon_0 q)^m q^{(2m-N+1)l} \\
&\quad = \sum_{m=0}^{\infty} \tilde{\chi}_m^+ (-\epsilon_0 q)^m \prod_{l=0}^{N-1}(1-q^{2(m-l)}) = \tilde{\chi}_N^+ \epsilon_0^N [N]!\, q^{\frac{N(N+1)}{2}+N}(q-q^{-1})^N + O(\epsilon_0^{N+1}),
\end{aligned}$$

where we have used the q-binomial theorem and $\prod_{l=0}^{N-1}(1-q^{2(m-l)}) = 0$ for $0 \leqslant m \leqslant N-1$. □

Proposition C.4 *Let q be the Nth primitive root of unity for odd N and the $2N$th primitive root of unity for odd N. The regular Bethe state $|R\rangle$ in the sector with $A^2 = q^L$ satisfies*

$$\frac{\varphi_+(x_0^+)^m}{m!} \frac{\varphi_+(x_1^-)^m}{m!} |R\rangle = \chi_{mN}^+ |R\rangle,$$

where $\chi_m^+ = \lim_{\varepsilon q^N \to 1} \tilde{\chi}_m^+$.

Proof We consider only the case $m = 1$. The case of general m is proved in a similar way by setting $M = R + mN$ and $n = mN$ in (C.2). From the lemma above, we have

$$(C_+)^N \left(\prod_{l \in Z_N} \frac{1}{m_+(z_l) n(z_l)} \tilde{B}_l \right) |R\rangle = \chi_N^+ ([N]!)^2 |R\rangle + O(\epsilon_0) + \text{off-diagonal terms},$$

which, in the limit $\Lambda \to \infty$, yields

$$\frac{(C_+)^N}{[N]!} \frac{(B_+)^N}{[N]!} |R\rangle = \tilde{\chi}_N^+ |R\rangle + \text{off-diagonal terms}.$$

In the sector with $A^2 = q^L$, the off-diagonal terms vanish in the limit $\varepsilon q^N \to 1$ [21]. □

By taking the limit $\varepsilon q^N \to 1$ in the definition of $\tilde{\chi}_m^+$ (C.3), we have

$$\frac{\prod_{j=1}^{N-1}\phi_+(\xi q^{2j-1})}{F_+(\xi q) F_+(\xi q^{-1})} = \sum_{m \geqslant 0} \chi_m^+ (-\xi)^m.$$

The numerator of the left-hand side is rewritten as

$$\prod_{j=1}^{N-1}\phi_+(\xi q^{2j-1}) = \prod_{n=1}^{L} \frac{1 - t_{p_n}^{-N}\xi^N q^{-N}}{1 - t_{p_n}^{-1}\xi q^{-1}}.$$

Then we obtain

$$\left(\prod_{n=1}^{L} \frac{1 - t_{p_n}^{-N}\xi^N q^{-N}}{1 - t_{p_n}^{-1}\xi q^{-1}} \right) \frac{1}{F_+(\xi q) F_+(\xi q^{-1})} = \sum_{m \geqslant 0} \chi_m^+ (-\xi)^m.$$

By taking the sum over $\tau = 0, 1, \ldots, N-1$ after the substitution $\xi \mapsto \xi q^{2\tau+1}$, we have

$$\sum_{\tau=0}^{N-1} \left(\prod_{n=1}^{L} \frac{1 - t_{p_n}^{-N} \xi^N}{1 - t_{p_n}^{-1} \xi q^{2\tau}} \right) \frac{1}{F_+(\xi q^{2\tau}) F_+(\xi q^{2\tau+2})}$$
$$= \sum_{\tau=0}^{N-1} \sum_{m \geqslant 0} \chi_m^+ (-\xi q)^m q^{2\tau m} = N \sum_{m \geqslant 0} \chi_{mN}^+ (-\xi^N)^m = N P_{\mathrm{D}}^+(\xi^N),$$

which proves Proposition 4.2.

We give a remark. One can derive Proposition 4.2 from the proof of the spin-1/2 inhomogeneous case through the fusion method [21]. However, we have presented the direct and straightforward approach here.

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